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ABSTRACT

This is part two of a two-part manual for teachers using SMSG junior high school text materials. A chapter-by-chapter commentary on the text is given as well as answers to all the exercises; a few chapters contain sample text questions. Chapter topics include: (1) real numbers; (2) similar triangles; (3) variation; (4) non-metric polyhedrons; (5) volumes and surface areas; (6) relative error; (7) permutations and combinations; and (8) probability. (MP)

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**SCHOOL
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MATHEMATICS FOR JUNIOR HIGH SCHOOL

Commentary for Teachers

VOLUME II (Part 2)

2

MATHEMATICS FOR JUNIOR HIGH SCHOOL

Commentary for Teachers

Volume II (Part 2)

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Mathematics for Junior High School

Volume II (Part 2)

Commentary for Teachers

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UNIT 5

REAL NUMBERS

This experimental unit is intended to give a first brief and intuitive approach to the real number system.

The spirit of the unit is in keeping with the approach outlined in the appendices of the Report of the Commission on Mathematics of the College Entrance Examination Board, Chapter 1, Algebra, Section 3, (see Bibliography which follows).

Principal emphases have been placed on three main points:

First, the irrational numbers do in fact exist, and are as "real" as the rational numbers with which the pupils now have some familiarity. The approach via infinite decimals is intended to strengthen this conviction.

Second, great stress is placed upon the one-to-one correspondence between the points of a line and the real numbers.

Third, the important property of density is treated in detail. It leads naturally to a development of the possibility of approximating an irrational number by rationals. The possibility of approximating rationals by irrationals is also worthy of note.

The development of decimal representations for rationals is intended to strengthen the pupil's grasp of the rational number system as well as to provide a plausible basis for the introduction of irrationals.

As a by-product of the treatment the pupil should be expected to have gained a fair mastery of the properties of the rational number system, the real number system and the irrational number system. He should be expected to state the properties in his own words. In particular the importance of density and closure should be continually emphasized.

The treatments of the real number system in the suggested bibilography will provide much worthwhile supplementary material.

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5-1. Answers

a) 38 37 39 n-1 n+1

b) 1, because n-1 = 0, which is not a counting number

c) Yes. No. The smallest is 1.

d) Closed under addition and multiplication. $5 + 4 = 9$, and 9 is a counting number. $5 \cdot 4 = 20$, and 20 is a counting number. Not closed under subtraction or division

$(4 - 5)$ is not a counting number.

$(4 \div 5)$ is not a counting number.

e) Addition and multiplication are defined, and subtraction and division are inverses of addition and multiplication respectively.

f. Two (9 and 10). 999. None. Not if they are consecutive counting numbers.

Class Discussion Exercises 5-1.

1. a) -5 -6 -4 n-1 n+1

   b) -1, because $-1 + 1 = 0$, and 0 is not a negative integer.

   c) No. Yes. -1 is the largest negative integer.

   d) Closed under addition. $-6 + (-5) = -11$. Not closed under subtraction, multiplication, and division.

   $-4 - (-6) = +2$; $(-4) \cdot (-6) = +24$; $(-4) \div (-6) = +\frac{2}{3}$.

2. Set of positive integers.

3. a) $\frac{23}{4}$ b) $\frac{12}{1}$ c) $\frac{43}{6}$ d) $-\frac{20}{3}$ e) $\frac{72}{100}$ f) $\frac{0}{6}$

   (or $\frac{0}{n}$, where n is any counting number. g) $\frac{7}{200}$ h) $\frac{47}{100}$

   i) $-\frac{5}{3}$ j) $\frac{69}{10}$.

4. a) $\frac{46}{8}$ b) $\frac{36}{3}$ c) $\frac{172}{24}$ d) $-\frac{40}{6}$ e) $\frac{36}{50}$ f) $\frac{0}{3}$

g) $\frac{35}{1000}$ h) $\frac{94}{200}$ i) $-\frac{25}{15}$ j) $\frac{690}{1000}$ (or any other which are reducible to the answers in Ex. 3.)

5. 12, 0.

6. $\frac{a}{b} = \frac{ca}{cb}$ where $b \neq 0$ and $c \neq 0$.

   $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$

7. a) $\frac{23}{4}$ b) $\frac{12}{1}$ c) $\frac{43}{6}$ d) $\frac{20}{3}$ e) $\frac{18}{25}$ f) 0

   g) $\frac{7}{200}$ h) $\frac{47}{100}$ i) $-\frac{5}{3}$ j) $\frac{69}{10}$

8. 1

9. a) $23 \div 4$ b) $12 \div 1$ c) $43 \div 6$ d) $(-20) \div 3$

   e) $72 \div 100$ f) $0 \div n$, where $n \neq 0$ g) $7 \div 200$

   h) $47 \div 100$ i) $(-5) \div 3$ j) $69 \div 10$

10. Yes

11. a) Closure under addition.
    b) Identity for addition.
    c) Identity for multiplication.
    d) Closure under multiplication.
    e) Distributive property of multiplication over addition.
    f) Additive inverse.
    g) Commutative property for multiplication.
    h) Commutative property for addition.
    i) Associative property for addition.
    j) Associative property for multiplication.

12. Closure under division does not hold, i.e., if a and b are integers $\frac{a}{b}$ is not necessarily an integer. Multiplicative

inverse does not hold, since if a is an integer, $\frac{1}{a}$ is not an integer.

Closure: $-6 + 5 = -1$, and -1 is an integer.

$-6 \cdot (+5) = -30$, and -30 is an integer.

$-6 - (+5) = -11$, and -11 is an integer.

Commutativity: $-6 + (+5) = (+5) + (-6) = -1.$

$-6 \cdot (+5) = (+5) \cdot (-6) = -30.$

Associativity: $-6 + [(+5) + (-4)] = -6 + (+1) = -5.$

$[-6 + (+5)] + (-4) = (-1) + (-4) = -5.$

$(-6) \cdot [(+5) \cdot (-4)] = (-6)(-20) = +120$

$[(-6) \cdot (+5)] \cdot (-4) = (-30) \cdot (-4) = +120$

Identities: $(-6) + 0 = -6$ $\qquad (-6) \cdot 1 = -6.$

Distributivity: $-6[(+5) + (-4)] = (-6)(+1) = -6$

$$\begin{aligned} -6[(+5) + (-4)] &= [(-6) \cdot (+5)] + [(-6) \cdot (-4)] \\ &= (-30) + (+24) \\ &= -6. \end{aligned}$$

Additive inverse: $-6 + [-6-6)] = -6 + (+6) = 0$

Order: $-6 < -4 \qquad +5 > -4.$

13. Closure under subtraction and division do not hold.

Identity element for addition (0) is not a positive integer.

Additive inverse and multiplicative inverse do not hold.

Closure: $(+6) + (+4) = +10$

$(+6) \cdot (+4) = +24$

Commutativity: $(+6) + (+4) = (+4) + (+6) = +10$

$(+6) \cdot (+4) = (+4) \cdot (+6) = +24$

Associativity: $(+6) + [(+4) + (+5)] = (+6) + (+9) = +15$

$[(+6) + (+4)] + (+5) = (+10) + (+5) = +15.$

Identity: $(+6) \cdot 1 = +6.$

Distributivity: $(+6)[(+4) + (+5)] = (+6) \cdot (+9) = +54$

$(+6)[(+4) + (+5)] = [(+6) \cdot (+4)] + [(+6) \cdot (+5)]$

$= (+24) + (+30)$

$= +54$

Order: $+6 > +5.$ $\quad +5 < +6$

14. Closure: $\frac{5}{3} + (-\frac{1}{3}) = +\frac{4}{3}$

$\frac{5}{3} - (-\frac{1}{3}) = +\frac{6}{3}$

$\frac{5}{3} \cdot (-\frac{1}{3}) = -\frac{5}{9}$

$\frac{5}{3} \div (-\frac{1}{3}) = -\frac{5}{1}$

Commutativity: $\frac{5}{3} + (-\frac{1}{3}) = (-\frac{1}{3}) + (+\frac{5}{3}) = +\frac{4}{3}$

$(\frac{5}{3}) \cdot (-\frac{1}{3}) = (-\frac{1}{3}) \cdot (\frac{5}{3}) = -\frac{5}{9}$

Associativity: $\frac{5}{3} + [(-\frac{1}{3}) + (\frac{3}{4})] = \frac{5}{3} + (+\frac{5}{12}) = \frac{25}{12}$

$[\frac{5}{3} + (-\frac{1}{3})] + \frac{3}{4} = \frac{4}{3} + \frac{3}{4} = \frac{25}{12}$

$\frac{5}{3} \cdot [(-\frac{1}{3}) \cdot \frac{3}{4}] = \frac{5}{3} \cdot (-\frac{1}{4}) = -\frac{5}{12}$

$[\frac{5}{3} \quad (-\frac{1}{3})] \quad \frac{3}{4} = (-\frac{5}{9}) \quad \frac{3}{4} = -\frac{15}{36} = -\frac{5}{12}.$

Identities: $-\frac{7}{8} + 0 = -\frac{7}{8}.$ $\quad (-\frac{7}{8}) \cdot 1 = -\frac{7}{8}$

Distributivity: $\frac{5}{3} \cdot (-\frac{1}{3} + \frac{3}{4}) = \frac{5}{3} \cdot (\frac{5}{12}) = \frac{25}{36}$

$\frac{5}{3} (-\frac{1}{3} + \frac{3}{4}) = (\frac{5}{3}) \cdot (-\frac{1}{3}) + (\frac{5}{3}) (\frac{3}{4})$

$= -\frac{5}{9} + (\frac{15}{12})$

$= -\frac{20}{36} + \frac{45}{36} = \frac{25}{36}$

Additive Inverse: $\frac{5}{3} + (-\frac{5}{3}) = 0$

Multiplicative Inverse: $(\frac{5}{3}) \cdot (\frac{3}{5}) = 1$

Order: $\frac{3}{4} > -\frac{1}{3}$. $\frac{3}{4} > \frac{5}{3}$.

15. $+\frac{7}{4}$. $-\frac{4}{7}$. Reciprocal.

Exercises 5-2a.

1.

2.

3. Yes. $-\frac{5}{5}, -\frac{6}{6}$. $-\frac{1}{5}, -\frac{1}{6}$. $\frac{0}{5}, \frac{0}{6}$. $\frac{1}{5}, \frac{1}{6}$.

4. Six new points, for $\frac{8}{7}, \frac{9}{7}, \frac{10}{7}, \frac{11}{7}, \frac{12}{7}, \frac{13}{7}$.

Six new points, for $\frac{22}{7}$ $\frac{23}{7}$ $\frac{24}{7}$ $\frac{25}{7}$ $\frac{26}{7}$ $\frac{27}{7}$.

5. Four new points.

6. a) $\frac{0}{3}$ names the same point as $\frac{0}{1}$.

b) $\frac{2}{4}$ names the same point as $\frac{1}{2}$.

c) In the row for sixths, $\frac{2}{6}$, $\frac{3}{6}$, and $\frac{4}{6}$ have already been named. Since 5 and 7 are prime numbers, points named by rational numbers with denominators 5 and 7 have not yet been named.

7. $\frac{1}{8} > 0$. $\frac{1}{8} = \frac{7}{56}$ and $\frac{1}{7} = \frac{8}{56}$. $\frac{8}{56} > \frac{7}{56}$.

$\frac{1}{7} = \frac{6}{42}$ and $\frac{1}{6} = \frac{7}{42}$. $\frac{7}{42} > \frac{6}{42}$.

$\frac{1}{6} = \frac{5}{30}$ and $\frac{1}{5} = \frac{6}{30}$. $\frac{6}{30} > \frac{5}{30}$.

$\frac{1}{5} = \frac{4}{20}$ and $\frac{1}{4} = \frac{5}{20}$. $\frac{5}{20} > \frac{4}{20}$.

$\frac{3}{4} = \frac{15}{20}$ and $\frac{4}{5} = \frac{16}{20}$. $\frac{16}{20} > \frac{15}{20}$.

$\frac{4}{5} = \frac{24}{30}$ and $\frac{5}{6} = \frac{25}{30}$. $\frac{25}{30} > \frac{24}{30}$

$\frac{5}{6} = \frac{35}{42}$ and $\frac{6}{7} = \frac{36}{42}$. $\frac{36}{42} > \frac{35}{42}$.

$\frac{6}{7} = \frac{48}{56}$ and $\frac{7}{8} = \frac{49}{56}$. $\frac{49}{56} > \frac{48}{56}$.

$\frac{1}{1} = \frac{8}{8}$ and $\frac{8}{8} > \frac{7}{8}$.

8. Six. Four. Ten.

9. Prime number. See answer to E 6.c.

10. $\frac{13}{84}$. $\frac{29}{70}$. $\frac{97}{112}$.

11 Twenty-two new points.

Exercises 5-2b.

1. No. There is no integer between -5 and -4.

2. Yes, 1 is the smallest. No largest positive integer.

3. No smallest negative integer. -1 is the largest.

4. No. No.

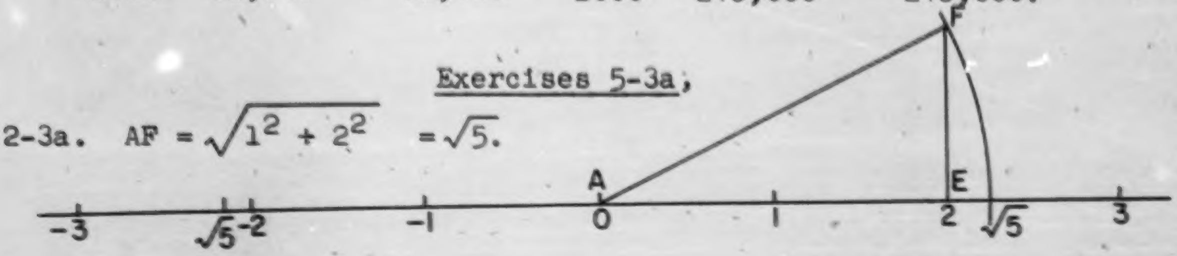
5. $\frac{1}{200}$. $\frac{1}{400}$.

6. Yes.

7. One such plan would be to use some other point than the halfway point. For example, the point $\frac{1}{3}$ of the way from $\frac{1}{1000}$ to $\frac{2}{1000}$ is $(\frac{1}{1000} + \frac{1}{3000})$ or $\frac{4}{3000}$. The next named points would be $(\frac{1}{1000} + \frac{1}{3000})$ or $\frac{4}{3000}$. The next named points would be $(\frac{1}{1000} + \frac{1}{9000})$ or $\frac{10}{9000}$; $(\frac{1}{1000} + \frac{1}{27,000})$ or $\frac{28}{27,000}$); $(\frac{1}{1000} + \frac{1}{81,000})$ or $\frac{82}{81,000}$; $(\frac{1}{1000} + \frac{1}{243,000})$ or $\frac{244}{243,000}$.

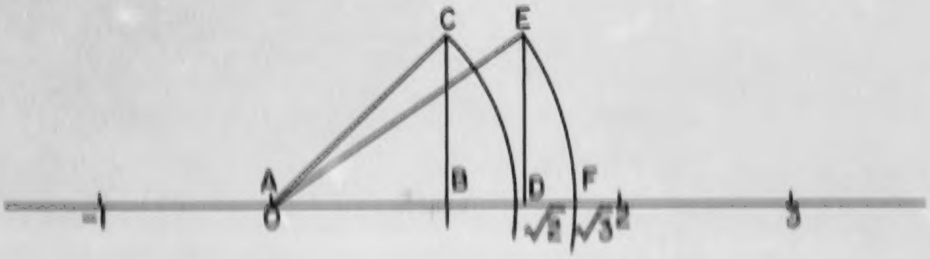
Exercises 5-3a.

2-3a. $AF = \sqrt{1^2 + 2^2} = \sqrt{5}$.



4. Irrational. There is no rational number $\frac{a}{b}$ such that $(\frac{a}{b})^2 = 5$.

*5.



$AB = 1$, $BC = 1$, so $AC = \sqrt{2}$. Make $AD = AC$, so point D represents $\sqrt{2}$.

Draw DE perpendicular to AD, $DE = 1$. Then $AE^2 = (\sqrt{2})^2 + 1^2$ and $AE = \sqrt{3}$. Make $AF = AE$ so point E corresponds to $\sqrt{3}$. Similarly $\sqrt{6}$ can be found by using a right triangle with sides of lengths $\sqrt{5}$ and 1; $\sqrt{7}$ by using a right triangle with sides of lengths $\sqrt{6}$ and 1.

6. Lay off the distance $\sqrt{2}$ twice to the right of 0; 3 times to the right of 0; three times to the left of 0.

7. Irrational.

*8. BRAINBUSTER. First show the following:

   1. If a number is divisible by 5, it can be expressed as $5 \cdot k$ where k is an integer.
   2. If a number is not divisible by 5, it can be expressed as $5n + R$, with $0 < R < 5$, R an integer. (Division Algorithm, Volume 1, Chapter 5.)
   3. If n is divisible by 5, then n^2 is divisible by 5. ($(5k)^2 = 25k^2 = 5(5k^2)$. $5k^2$ is an integer by the closure property of integers under multiplication).

4. If n is not divisible by 5, than n^2 is not divisible by 5.

   If n is not divisible by 5, it may be expressed as $5k + 1$, or $5k + 2$, or $5k + 3$, or $5k + 4$.

   $(5k + 1)^2 = 25k^2 + 10k + 1 = 5(5k^2 + 2k) + 1$. ∴ it is not divisible by 5.

   $(5k + 2)^2 = 25k^2 + 20k + 4 = 5(5k^2 + 4k) + 4$. ∴ it is not divisible by 5.

   $(5k + 3)^2 = 25k^2 + 30k + 9 = 5(5k^2 + 6k + 1) + 4$. ∴ it is not divisible by 5.

   $(5k + 4)^2 = 25k^2 + 40k + 16 = 5(5k^2 + 8k + 3) + 1$. ∴ it is not divisible by 5.

5. If n is divisible by 5, and $\sqrt{n}$ is an integer, then $\sqrt{n}$ is divisible by 5.

   Use an indirect argument.

   Assume $\sqrt{n}$ is not divisible by 5. Then by (4) above, n is not divisible by 5. But n is divisible by 5. Therefore $\sqrt{n}$ is divisible by 5.

   Now the proof follows the same form as the proof that 2 is not a rational number.

   1. Assume $\sqrt{5}$ is a rational number. Then $\sqrt{5} = \frac{p}{q}$, p and q integers and $q \neq 0$. Let $\frac{p}{q}$ be the simplest expression for $\sqrt{5}$, so p and q have no common factors other than 1.
   2. $5 = \frac{p^2}{q^2}$.
   3. $\therefore 5q^2 = p^2$
   4. q^2 is an integer
   5. $\therefore p^2$ is divisible by 5.

6. $\therefore p$ is divisible by 5.

   Let $p = 5k$, k an integer.

7. $5q^2 = (5k)^2$ or $25k^2$

8. $q^2 = 5k^2$

9. k^2 is an integer.

10. $\therefore q^2$ is divisible by 5.

11. $\therefore q^2$ is divisible by 5.

12. By statements 6 and 11, both p and q have the factor 5. But this contradicts the condition in Statement 1. Therefore the assumption that $\sqrt{5}$ is a rational number is false

13. $\therefore \sqrt{5}$ is not a rational number.

Exercises 5-4.

1. $5 < \sqrt{30} < 6$.
2. $9 < \sqrt{89} < 10$.
3. $15 < \sqrt{253} < 16$.
4. $65 < \sqrt{4280} < 66$.
5. $96 < \sqrt{9315} < 97$.
6. a) 3
   b) 2.999824
   c) 3.003289
   d) .000176 .003289
   e) 1.732
7. $(1.73)^2 = 2.9929$ $(1.74)^2 = 3.0276$ 1.73 is the better.
8. $(3.87)^2 = 14.9769$ $(3.88)^2 = 15.0544$ 3.87 is the better.
9. $(25.2)^2 = 635.04$ $(25.3)^2 = 640.09$ 25.2 is the better.
10. 3.2
11. 12.2

12. 14.9

*13. $m^2 = 100$

*14. $m^2 = 22,201$

Exercises 5-5.

1. $.378378378\overline{378}\ldots$

2. $.428571\overline{428571}\ldots$

3. $2.2500000\overline{0}\ldots$

4. $.02439\overline{02439}\ldots$

5. $.01210121\overline{0121}\ldots$

6. $.0925925\overline{925}\ldots$

7. $1.024000000\overline{0}\ldots$

8. $.0857142\overline{857142}\ldots$

9. $.012195\overline{12195}\ldots$

*10. $.\overline{0588235294117647}\ldots$

11. a) $\frac{9}{4}$ and $\frac{128}{125}$.

b) $4 = 2^2$ $124 = 5^3$. They are powers of 2 and 5.

12. a) Decimal for $\frac{3}{7}$ is 5 times the decimal for $\frac{3}{35}$. 35 is $5 \cdot 7$.

b) Decimal for $\frac{1}{41}$ is 2 times the decimal for $\frac{1}{82}$, because 82 is $2 \cdot 41$, and $\frac{1}{41}$ is therefore $2 \cdot \frac{1}{82}$.

Exercises 5-6.

1. $\frac{1}{11}$.

2. $\frac{1}{9}$

3. $\frac{1}{18}$ ($m = .05\overline{5}\ldots$ $100m = 5.5\overline{5}\ldots$ $10m = .5\overline{5}\ldots$ $90m = 5$)

4. $\frac{1}{30}$ ($m = .03\overline{3}\ldots$ $100m = 3.\overline{3}\ldots$ $10m = .3\overline{3}\ldots$ $90m = 3$)

5. $\frac{123}{999} = \frac{41}{333}$

6. $\frac{1625}{10,000} = \frac{13}{80}$

7. $\frac{1}{6}$

8. a) $32 = 2^5$ $100 = 2^2.5^2$ $14 = 2\cdot 7$ $26 = 5^2$

   $50 = 2\cdot 5^2$ $9 = 3^2$ $6 = 2\cdot 3$

   b) $\frac{7}{32}$ $\frac{47}{100}$ $\frac{3}{25}$ $\frac{17}{50}$

Exercises 5-7.

1. a) 1.372 1.379 1.385 1.493 5.468

   b) -9.426 -5.630 -2.765 -2.763 -2.761

   c) -.15475 .15463 .15467 .15475 .15598

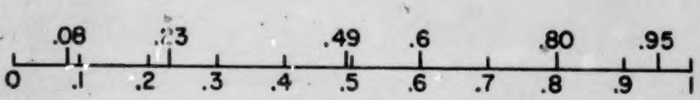
2. a) None

   b) None

   c) .15463 .15467 .15475 .15598

   d) .15463 .15467 .15475 .15598

   e) .15463 .15467 .15475



Exercises 5-9.

1. Rational: a) $.231\overline{231}\ldots$ Irrational: b) 23123112311123...

   c) $\frac{3\sqrt{2}}{7\sqrt{2}} = \frac{3}{7}$ d) $\sqrt{7}$

   e) $.78\overline{342}\ldots$ f) $\frac{\pi}{2}$

   i) .75000 g) $\frac{3}{4}\sqrt{6}$

   j) $\frac{58}{11}$ h) $9 - \sqrt{3}$

2. a) $.231\overline{231} = \frac{231}{999}$

c) $\frac{3\sqrt{2}}{7\sqrt{2}} = \frac{3}{7} = .\overline{428571}...$

e) $.78\overline{342}... = \frac{78264}{99,900} = \frac{2174}{2775}$

i) $.75000 = \frac{3}{4}$

j) $\frac{58}{11} = 5.27\overline{27}...$

3. b) .231231123111123... = .23

d) $\sqrt{7} = 2.65$

f) $\frac{\pi}{2} = 1.57$

g) $\frac{3}{4}\sqrt{6} = \frac{3\cdot(2.45)}{4} = \frac{7.35}{4} = 1.84$

h) $9 - \sqrt{3} = 9 - 1.73 = 7.27$

4. a) $.45632\overline{0}...$ $8.27\overline{0}...$ $.75\overline{0}...$

b) $.45\overline{45}...$ $12.762\overline{762}...$ $.063\overline{63}...$

c) .450450045... .123112311123... .565665666...

5. a) $2.375\overline{375}...$

b) 2.370370037000...

6. a) $.34\overline{4}...$

b) .345634456344456...

7. a) 67.[illegible]...

b) 67.28292020020002...

UNIT 6

SIMILAR TRIANGLES

General Remarks

This development of properties of similar triangles is different from that which appears in many texts. The essential difference is that first considerations are based on the assumption that the graph of equations of the type $y = ax$, where "a" is any positive real number, is a straight line. This of course has not been proved in earlier chapters. In these chapters, pupils are asked to locate points on graph paper and to observe that these points (in appropriate examples) appear to lie on a straight line. The decision to start the discussion of similar triangles on this assumption was made for a number of reasons not the least important of which were to relate this content much more closely to previous material and to teach more about properties of graphs.

The pupil is led to an understanding of the property that if two triangles have corresponding angles equal in measure then corresponding pairs of sides are proportional, first through very special cases of right triangles, then with right triangles in general, and finally to other types of triangles. In each section the limitations of the arguments given (these limitations being due to the very special cases considered) are carefully identified. If these limitations are emphasized the teacher has an opportunity to extend the pupil's understanding of proof and some of the pitfalls which may be found in arguments that may be represented as proofs, in geometry and in other areas of thought.

The authors consider the objectives of this experimental unit to be for the pupil to acquire some understanding of and some experience with:

1. The constant ratio of the y-coordinate to the x-coordinate of points on graphs of the type y = ax where "a" is a positive real number, not zero. (Section 1)
2. The property that if one acute angle of a right triangle is equal in measure to an acute angle of another right triangle, the corresponding sides are proportional. (Section 2)
3. The proof of the similarity property in special cases and possibilities of extension of the proof to more general cases. (Section 3)
4. The impossibility of using simple methods in geometric problems where the ratio is an irrational number. (Section 3)
5. The fact that the coordinate axes do not have to be thought of as vertical and horizontal for y and x respectively. (Section 4)
6. The trigonometric ratios, sine of an angle and tangent of an angle. (Section 5)
7. The slope of a line. (Section 6)
8. Reading a mathematical table. (Section 7)
9. The correspondence of similarity for triangles. (Section 8)
10. Assumptions regarding similar triangles, which are not right triangles (without proof or inductive development)(Section 9)
11. Applications of similarity to indirect measurement. (Section 9)

The unit also provides much additional experience with ratio and proportion.

Careful deductive proofs of similarity properties are among the most difficult proofs in elementary mathematics. It does not seem obvious that two triangles necessarily exist for which the measure of angles are equal and the corresponding sides proportional. The slow-moving development for special cases of right triangles is intended to lead the student to the idea that this is a reasonable property to expect.

The teacher will recognize that scale drawings are a simple and common application of the similarity concept and may wish to provide supplementary materials for this chapter on this topic.

If any part were to be omitted it is suggested that it might be the last two sections.

This unit assumes that the pupil will have studied Unit 4 of the experimental course. Also some knowledge of properties associated with parallel lines and transversals is assumed. It is pointed out in the teaching suggestions which follow that previous experience in graphing first degree equations, the graphs of which pass through the origin is also very desirable. If pupils have studied Unit 5 on Real Numbers they should have little difficulty in understanding the references to rational and irrational numbers. These references, however, do not play a critical role in this development.

6-1. The Graph of $y = 2x$ (First Quadrant)

Pupils should be skillful in plotting points and drawing line graphs by this time. Their understanding of graphs may not include the idea that if the coordinates satisfy the equation the point is on the graph and if a point is on the graph, its coordinates satisfy the equation. This idea is expressed in the pupil text but may need further discussion. Also, the statement is clearly put forth that we have not proved that the graph of y = ax for "a" a positive real number is a straight line. We assume that this is true through the chapter.

The exercises are designed to emphasize the coordinates of a point on a graph of y = 2x, may be written as (x,2x).

Answers to Exercises 6-1a

1. x 01234.....

   y = 2x 02468 ....

   Only positive numbers and zero are suitable for x.

   y cannot be negative since 2x cannot be negative in the part of the graph in this discussion.

2. The curve is a line like the one of which $\overline{OI}$ is a segment in Figure 6-1.

3. If the x-coordinate of (P) on the graph is a, then from the equation y = 2x, the y-coordinate is 2a.

4. (a) OA is the measure of the distance of S from the x-axis and hence is the x-coordinate.

   (b) SA is the measure of the distance of S from the x-axis and hence is the y-coordinate.

   (c) From Problem 3 the y-coordinate of S is twice the x-coordinate. SA = 2(OA)

   (d) From (c) SA/OA = 2. From y = 2x, y/x = 2. Then SA/OA = y/x = 2.

5. (a) T is on the line. Hence its coordinates satisfy the equation of the line and y' = 2x'.

(b) OB is the measure of the distance of the point T from the x-axis. Then OB = x'. BT is the measure of the distance of the point T from the x-axis. Then BT = y' also y' = 2x'; BT/OB = y'/x' = 2.

Answers to Exercises 6-1b

1. OB/OA = BT/SA
2. $\angle$ASO and $\angle$BTO, $\angle$SAD and $\angle$TBO, $\angle$SOA and $\angle$TOB.
3. yes
4. yes
5. (x', 5x')
6. (x, 1/3x)

6-2. A Special Set of Right Triangles

This section deals with the special set of triangles in which the graph of y = 2x is the line of which one side of the triangle is a segment. The second restriction is that the length OB is 2(OA).

Congruent triangles are used to establish that the hypotenuse of the larger triangle is twice the hypotenuse of the smaller triangle.

Thus in this special set of triangles, the ratio of the longer leg to the shorter leg of the same triangle is 2, and the ratio of a side of the larger triangle to the corresponding side of the smaller triangle is 2.

Answers to Exercises 6-2

1. (a) $\overline{AS}$ and $\overline{BT}$ are parallel. They are both perpendicular to the x-axis.

   (b) The coordinates of T are the measures of the distances from T to the axis x' = OB and y' = BT.

(c) From $y = 2x$ BT = 2(OB).

(d) From the construction of OB, OB = 2(OA) and then 2(OB) = 4(OA).

(e) BT = 4(OA)

2. (a) OB = 2 (b) SA = 2 (c) BT = 4

3. (a) $\triangle$OAS is a right triangle. SA is perpendicular to OA. $\triangle$OBT is a right triangle. BT is perpendicular to OB.

(b) $\angle$SAD, and $\angle$TBO are right angles.

(c) $\angle$SOA is in both triangles.

(d) The sum of the angle measures is 180.

4. Since AS is parallel to BT, m(LOSA) = m(LOTB) since corresponding angles of parallel lines are congruent.

5. (a) $\overline{AB}$ and $\overline{SK}$ are perpendicular to BT and are parallel.

(b) $\overline{AS}$ and $\overline{BK}$ are perpendicular to AB and are parallel.

(c) ABKS is a parallelogram. (It can be shown as a parallelogram.)

(d) SK = AB since opposite sides of a parallelogram are equal.

6. AB = 1 so SK = 1 from 5(d).

7. AS = BK

AS = 2

BK = 2

BT = 4

KT = BT - BK

KT = 2

AS = KT

8. $\triangle$OAS and $\triangle$SKT are congruent since 2 sides and the included angle of one are congruent to 2 sides and the

included angle of the other.

9. OS = ST because corresponding sides of congruent triangles are congruent.

10. $OT = OS + ST$

    $OT = 2(OS)$

    $OT/OB = 2$

11. OT and OS

    OB and OA

    BT and AS

6-3. The Graph of $y = ax$ (First Quadrant, a is a Positive Real Number)

The sets of right triangles under consideration are extended here to those related to the graph of $y = 3x$; then the sets of right triangles related to $y = 7/3$; and finally to the sets related to $y = ax$.

As long as the ratio of the sides on the x-axis is kept at 2, congruent triangles are readily available to establish the ratio of the sides opposite the right angles. When the ratio of the side of one triangle to the corresponding side of the other is a rational number, congruent triangles may still be used. Exercises 10 and 11 are intended to encourage pupils to try this method where the ratio of the sides on the x-axis is no longer 2.

Answers to Exercises 6-3a

1. (a) (b) See Figure 6-3a

   (c) $KR = 3(OR)$ $KR = 3$ $(x, 3x)$ or (OR, KR) or $(OR, 3(OR)$

   (d) See Figure 6-3a

   (e) $\angle S = 3(OS)$ $\angle S = 6$

(f) $OS/OR = 2$ $SL/RR = 2$

2. (a) $OL = 5m$ $OK = 2.5$ $OL/OK = 5/2.5 = 2$

(b) KMSR is a parallelogram since its opposite sides are perpendicular to the same line and hence are parallel. $KM = RS$ since opposite sides of a parallelogram are equal. In $\triangle KML$ and $\triangle ORK$

$KM = OR$ since $RS = OR$

$KR = MS$, measures of these segments are equal numbers.

$m(\angle OKR) = m(\angle OLS)$ corresponding angles of parallel lines are equal.

$\triangle KML$ and $\triangle ORK$ are congruent by 2 sides and included angle property.

$OK = KL$ corresponding sides of congruent triangles are congruent.

$OL = 2OK$ and $OL/OK = 2$.

3. (a) 3

(b) 2

(c) The ratio of the longer leg to the shorter of a right triangle with its hypotenuse on $y = 3x$ is 3. If the hypotenuse of the triangle is on the graph of $y = 2x$, the ratio is two. It appears as if this ratio is a for $y = ax$.

(d) Ratio of leg of larger triangle to leg of shorter triangle is 2.

4. (a) See Figure 6-3b.

(b) RA and BT are perpendicular to AB and measure of each angle in degrees is 90. Yes.

(c) RA = 7/3

(d) BT = 14/3

(e) 7/3

(f) 7/3

(g) The ratios equal the same number.

(h) No. Ratio which do depend on measuring of OA and OB are OA/OB, RA/BT, OR/OT.

5. The ratios obtained through the use of measurements should be equal.

6. In 6-1 it was proved that if $y/x = y'/x'$ then $y/y' = x/x'$.

Answers to Exercises 6-3b

1. (x_1, ax_1)
2. $y_1 = O_1P_1$ and $x_1 = OQ_1$ so $y_1/x_1 = Q_1P_1/OQ_1$
3. $y_1 = ax_1$ $O_1P_1/OQ_1 = ax_1/x_1$ so $O_1Q_1/OQ_1 = a$
4. P_2 is on $y = ax$. Therefore, $y_2 = ax_2$.
5. P_2Q_2 is the y-coordinate of P_2 and must equal ax_2.
   OQ_2 is the x-coordinate of P_2 and must equal x_2.
   $P_2Q_2/OQ_2 = ax_2/x_2$, $P_2Q_2/OQ_2 = a$
6. $OP_2 = 73$ cm $OQ_2 = 62$ cm $P_2Q_2 = 38$
   $OP_1 = 49$ cm $OQ_1 = 42$ cm $P_1Q_1 = 26$
   $OP_2/OP_1 = 1.49$ $OQ_2/OQ_1 = 1.47$ $P_2Q_2/P_1Q_1 = 1.46$
7. $y = ax$, $y/x = a$
8. $y/x = Q_1P_1/OQ_1 = P_2Q_2/OQ_2$ since each ratio is "a".
9. It is necessary for $OQ = OQ_2$ in order to have congruent triangles.
10. Show triangle OCP, CFK', and AFH are congruent.
11. Show that the small right triangles on $\overline{OC}$ are congruent.

12. No. In this case we could not determine the number of congruent triangles to use.

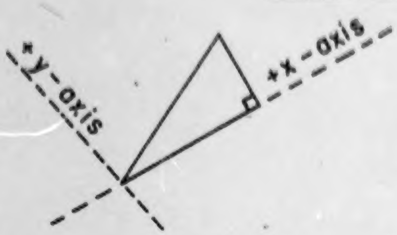
6-4. The Similarity Property for Right Triangles

In this section we are trying out the idea that coordinate axes are not necessarily vertical and horizontal. They may be placed to fit right triangles shown in any position. Then the graph method may be applied and it follows that the properties observed in the special cases of Sections 1-2-3 apply for any right triangles.

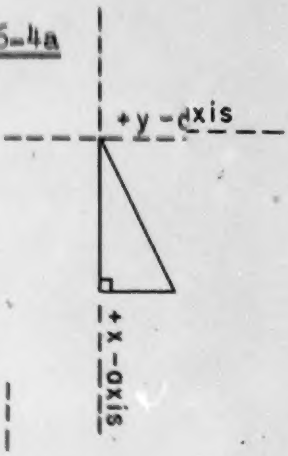
Show pupils that the triangles may be moved, if this is preferred. The phrase "by a rigid motion" means that the triangle is merely transported. The only thing changed is the location of the triangle. The similarity property holds for all right triangles.

Answers to Exercises 6-4a

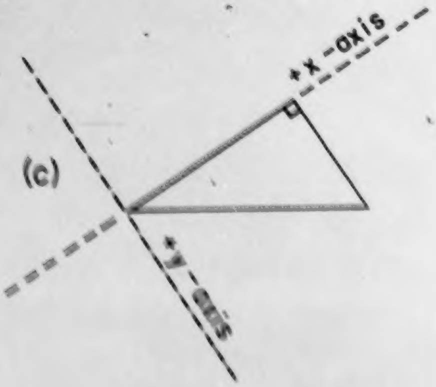
(a)



(b)



(c)



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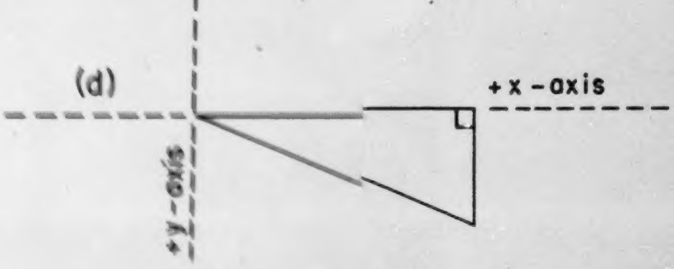
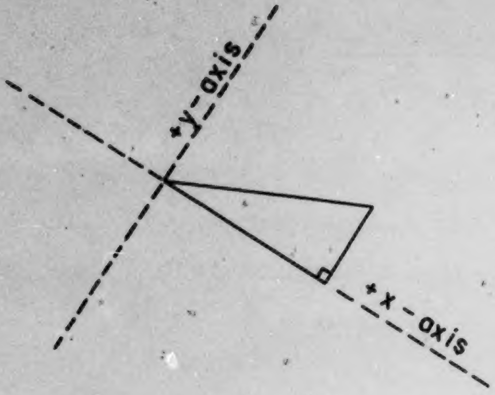
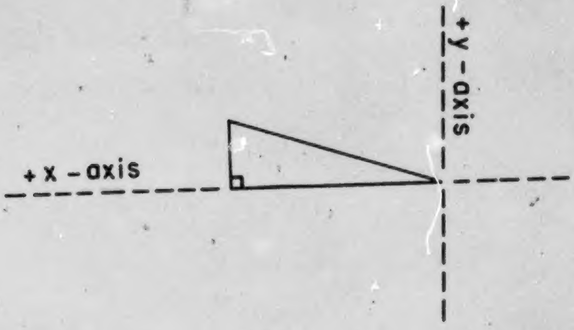






Answers to Exercises 6-4b

1. ---
2. In $y = ax$, "a" is the ratio.
3. (a) 9/16 and 25/44
   (b) 13/19 and 20/30
   (c) 32/36 and 18/20
   (d) 37/24 and 28/18
   (e) 13/18 and 39/53

Answers to Exercises 6-4c

1. $AC = 5$, $A'C' = 25$ $AC/A'C' = 1/5$
2. $AC = 13$, $A'C' = 26$ $AC/A'C' = 1/2$
3. $AC = 17$, $A'C' = 51$ $AC/A'C' = 1/3$
4. $AC = 10$, $A'C' = 15$ $AC/A'C' = 2/3$
5. $AC = \sqrt{5}$, $A'C' = \sqrt{20}$ $AC/A'C' = 1/2$

6-5. Sines and Tangents

The introduction to the sine and tangent is accomplished by using the graph of a line in the first quadrant and forming a series of right triangles all of which are shown to be similar. The Pythagorean property shows the relation of r to x and y. In the exercises metric coordinate paper is used to find the sine and tangent of given angles. The unit circle is introduced. (first quadrant)

Answers to Exercises 6-5

1. (a) Triangle sides are 3,4,5,
sine, 4/5 tan, 4/3 angle measure is 53°
(b) Triangle sides are 5,12,13
sine, 5/13 tan, 5/12 angle measure is 23°
(c) Sine, 1/2 tan, 1 angle measure 45°

3. (a) Triangle sides are 3,4,5
sine, 3/5 tan, 3/4
(b) Triangle sides are $2, 5, \sqrt{29}$
sine, $5/\sqrt{29}$ tan, 5/2
(c) Triangle sides are $1, \sqrt{3}, 2$
sine, $\sqrt{3}/2$ tan, $\sqrt{3}$

4. (a) sine $\angle BAC = 1/\sqrt{10}$ sine $\angle CBA = 3/\sqrt{10}$
tan $\angle BAC = 1/3$ tan $\angle CBA = 3$
(b) sine $\angle FED = 4/\sqrt{41}$ sine $\angle FDE = 5/\sqrt{41}$
tan $\angle FED = 4/5$ tan $\angle FDE = 5/4$

5. (a) sine $40^\circ \approx .6428$
tan $40^\circ \approx .8391$
(b) sine $70^\circ \approx .9397$
tan $70^\circ \approx 2.7475$

6. (a) 14 feet
(b) 35 feet

7. (a) 0.6 0.5774
(b) 0.8 0.7813
(c) 1.2 1.192
(d) 1.9 1.881

8. 64 degrees

9. (a) 0.5 0.5000

(b)	0.6	0.6157
(c)	0.8	0.7660
(d)	0.9	0.8829

6-6. Slope of a Line

The word "slope" is introduced and the connection between a in $y = ax$ and the slope is investigated by an inspection process. It is important for pupils to do the cutting and folding of paper described.

Have the pupils discuss the graph of $y = x$ and the slope of this line as an additional exercise.

Answers to Exercises 6-6

1. Slope of $y = 2/3x$ is $2/3$

   Slope of $y = 1/4x$ is $1/4$

   Slope of $y = 4x$ is 4

2. (a) $3/1 = 3$

   (b) $3/2$

   (c) $5/\frac{1}{2} = 10/1 = 10$

   (d) $6/1\frac{1}{2} = 12/3 = 4/1 = 4$

3. $5/3$ if x is 3 and y is 5

   $3/5$ if x is 5 and y is 3

4. $10/5280 = 1/528$

5. sine, $1/\sqrt{2}$ (sometimes written $\sqrt{2}/2$)

   tangent, 1

6. (a) $\dfrac{\sin 60^\circ}{\sin 30^\circ} = \dfrac{\sqrt{3}/2}{1/2} = \dfrac{\sqrt{3}}{1} = \sqrt{3}$

   (b) $\dfrac{\tan 60^\circ}{\tan 30^\circ} = \dfrac{\sqrt{3}/1}{1/\sqrt{3}} = \sqrt{3} \cdot \sqrt{3} = 3$

   (c) In this problem we see that the slope of a line at 60° with the horizontal is 3 times that at 30° with the horizontal.

7. The measure of the angle formed by the graph of $y = 7x$ and $y = 8x$ is smaller than the measure of the angle formed by lines of slope 1 and 2.

6-7. Reading a Table

The measurement and computation of sines and tangents in the preceding sections were included to prepare pupils for this work with tables of values. Pupils usually understand readily how a table is organized and how to use it but some class discussion is often helpful. Be sure the pupils understand how to deal with angles greater in measurement than 45 degrees.

The form used in writing the steps of a solution on page is a good one to follow.

Answers to Exercises 6-7

1. (a) 0.1737 (f) 0.8391
   (b) 0.1763 (g) 1.1918
   (c) 0.6561 (h) 1.7321
   (d) 0.8910 (i) 2.7475
   (e) 0.9903 (j) 0.9994
2. A. yes
   B. yes
   C. yes
   D. yes
   E. yes
3. A. Always between 0 and (no upper limit).
   B. Same property as stated.

C. Tangent of angles less than 45° are less than 1.

D. Same property as stated.

E. The difference between the tangents of two consecutive angles is smaller for smaller consecutive angles than larger ones.

4. Division by zero is meaningless.

5. (a) 52.99 (c) 0.258201

(b) 89.9586 (d) 0.283565

6. (a) AD = 86.603

(b) BC = 138.5907809

In 5 and 6, these answers have not been rounded. The teacher should round according to class practice established.

6-8. Proportionality in Geometric Figures

Pupils should be encouraged to name many examples of similar geometric figures and of applications of the principles. They might mention photograph enlargements, dress patterns, projection of movies on a screen, scale drawings.

Notation and vocabulary used in similarity are introduced. This follows closely the method used in the SMSG text for 10th grade.

Answers to Exercises 6-8

1. (b) segments (c) rectangles as given (e) cubes

2. Yes. Corresponding angles are equal and the ratio of corresponding sides is 1.

3. (a) No. (b) Yes.

4. (a) Yes. (b) Corresponding vertices are at the F pts.,

A pts., and at O (origin). OA_1 corresponds to OA_2

A_1P_1 corresponds to A_2P_2

$O P_1$ corresponds to $O P_2$

5. $b'' = 6$; $c'' = 6$.
6. Corresponding angles are equal.
7. $a = \frac{8}{7}$; $b'' = \frac{21}{4}$; $c = \frac{16}{7}$; $d = \frac{20}{7}$
8. 10 11 12.1 13.31

30 33 36.3 39.93

6-9. The Similarity Property for Triangles

The similarity property for triangles in general is given here but the proof is too difficult for this level. The use of the property provides opportunity for practice in basic operations with rational numbers and includes much additional experience with ratio and properties.

Answers to Exercises 6-9

1. (a) $s' = \frac{9}{2}$ $t' = 6$

   (b) $r' = 15$ $s' = 20$

   (c) $r' = t' = 30$ $s' = 20$

   (d) $r = s = 14$

   (e) $r = 5\ 3$, $t = 10$
2. $k = \frac{16}{3}$; $m' = 3$; $\ell' = 18$; $m = 9$
3. $66\frac{2}{3}$ ft., $66\frac{2}{3}$ ft., $33\frac{1}{3}$ ft.
4. AB + BC + CA 3.5 + 7 + 9.6 or 20.1
5. (a) The ratio of perimeter of triangle ABC to the perimeter of triangle A'B'B' is $\frac{1}{2}$.

   (b) The ratio of the areas is $\frac{1}{4}$.

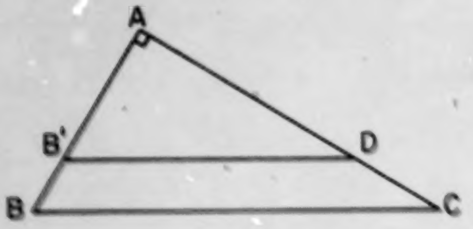
6. (a) Ratio of perimeters is $\frac{1}{3}$. Ratio of areas is $\frac{1}{9}$.

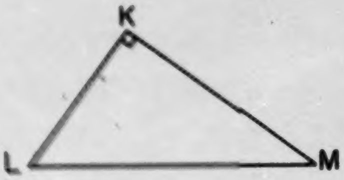
   (b) Ratio of perimeters is $\frac{1}{5}$. Ratio of areas is $\frac{1}{25}$.

7. The ratio of the perimeters of 2 similar triangles is equal to the ratio of a pair of corresponding sides.

8. The ratio of the areas of 2 similar triangles is equal to the square of the ratio of a pair of corresponding sides.

9. If the corresponding sides of two right triangles are proportional, then the triangles are similar.





It is given that the corresponding sides of triangles ABC and KLM are proportional. Therefore,

$$\frac{AB}{KL} = \frac{BC}{LM} = \frac{AC}{KM}.$$

Let the constant of proportionality be a/b.

Locate B' on AB so that AB' = KL. It is assumed AB > KL.

Since $\frac{KL}{AB} = \frac{a}{b}$, $\frac{AB'}{AB} = \frac{a}{b}$.

Draw through B' segment B'D so that B'D is parallel to BC.

Then

$m(\angle AB'D) = m(\angle ABC)$, by construction,

$m(\angle ADB') = m(\angle ACB)$, by corresponding angles,

Triangle AB'D and ABC are similar, (measure of angles are equal triangles are similar)

Now if it can be shown that triangles AB'D and KLM are congruent, we will know that the measures of corresponding angles of KLM and ABC are equal and the property will have been proved.

Investigating AB'D and KLM it is seen that

$AB' = KL$ and $m(\angle BAC) = m(\angle KLM)$ (right angles)

(1) Also $\frac{AB}{KL} = \frac{AC}{KM}$, given

(2) but $\frac{AB}{AB'} = \frac{AC}{AD}$, (triangles AB'D and ABC are similar)

(3) and $\frac{AB}{KL} = \frac{AC}{AD}$, (replacing AB' by KL)

Now if we compare (1) and (3) we see that they are the same except that AD is in (3) and KM is in the same position in (1).

We conclude that $AD = KM$.

Hence triangles AB'D and KLM are congruent.

The measures of corresponding angles of ABC and KLM are equal; and the property is proved.

UNIT 7

VARIATION

Overview

This chapter introduces the terminology used in connection with direct and inverse variation and gives several examples of applied situations in which these types of variation occur.

It should be emphasized that the statement " y varies directly as x " is the translation into English of the mathematical statement " $y = kx$ for some constant k ". That is, " y varies directly as x " gives us an equation of a certain type. It must be noted however, that unless additional information is given, we cannot find the numerical value of one of the variables which corresponds to a given numerical value of the other variable. We know that k is some constant, but we do not know the value of it. Usually one pair of corresponding values for x and y is given and, from this information, the constant k can be found. This gives us a specific equation relation x and y , and the value of y which corresponds to any given value of x can then be found.

The discussion in the last paragraph could be paraphrased in connection with the statement " y varies inversely as x " or " y varies directly as the square of x ".

Throughout the chapter, k is a positive number, but the pupils may be interested in an applied situation for which k is negative. A motion problem in which the direction of motion is important provides the following illustration: Consider an ordinary number line. Call the direction to the right "the positive direction"; and the direction to the left "the negative direction". Then if an object (work, butterfly, automobile, point) moves to the left along this line 5 feet every second, it is natural to say that its velocity is -5 feet per second. The equation $d = vt$ becomes in this case $d = -5t$. As usual, d is the distance of the object from its position at the start. (the term "start" refers to the time, $t = 0$, when our observations started. It does not imply that the motion started then--the object may have been moving for some time before we observed it.) Some number pairs which satisfy the equation $d = -5t$, and their interpretations, are given below.

(2, -10): Two seconds after the start, the object was 10 inches to the left of its starting position.

(3, -15): Three seconds after the start, the object was 15 inches to the left of its starting position.

(-4, 20): Four seconds before the start of our observations, the object was 20 inches to the right of the position it occupied at the time $t = 0$. (This assumes that the object was moving along the line before we observed it.)

7-1. Direct Variation

Direct variation is introduced by means of the automatic registers of number of gallons and cost on a gasoline pump at a filling station. The first exercises focus attention upon situations where direct variation occurs. The questions emphasize the idea that change in one variable is accompanied by a proportional change in the other. They provide a basis for class discussion and are designed to help the pupil derive the general equation. Encourage pupils to state in words what the equation says in symbols, before they read the printed definition.

The missing values in the Table in Section 7-1 are:

No. Gals.	1	2	3	4	5	6	7	8	9	10	11
Cost in $	.32	.64	.96	1.28	1.60	1.92	2.24	2.56	2.88	3.20	3.52

Answers to Exercises 7-1a

1. $t = 32n$, $d = .32n$

   t is multiplied by 5.

2. $c = 33.9g$ g is doubled.
3. $c = 7n$ No; as c increases, so does n increase.
4. $d = 2n$ No; n and d both increase or both decrease.
5. $i = 12f$ As f decreases, i decreases.
6. $p = 4s$ Yes; as p decreases s decreases in the same ratio.

The problems in Exercises 7-1b show several ways in which the number pairs may be listed for convenience in interpreting and noting pattern. Be sure pupils understand that in p/s, s cannot be zero.

In Problem 6, the equation is difficult at this stage. It is not intended to disturb pupils who cannot write it.

The plotting of points is familiar and pupils should have no trouble graphing lines. In setting up scales, you may wish to point out that a scale of one for each space is sometimes inappropriate when very large numbers arise. The scales may be altered for convenience, but is recommended that each space be assigned such values as 2, 5, 10, 20, 50, 100, etc. Quarter inch graph paper is quite satisfactory for work like this.

Proportion has been explained before. Here pupils apply the ideas.

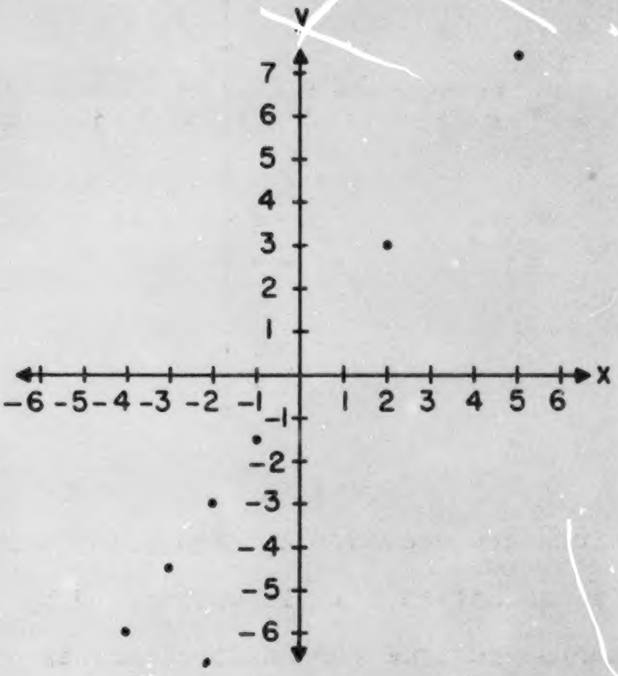
All examples use positive values of k . For pupils who are interested in what happens in cases where k is negative, a few leading questions are asked. A few specific examples, such as $y = -2x$, or $y = -5x$ should help.

In $y = kx$ where $k < o$, when x is multiplied by 10, y is also multiplied by 10 (if $x_2 = 10x_1$, then $kx_2 = 10kx_1$). When x increases, y decreases (if $x_2 > x_1$, and $k < 0$, then $kx_2 < kx_1$). If the class cannot handle this, you may wish to skip it entirely.

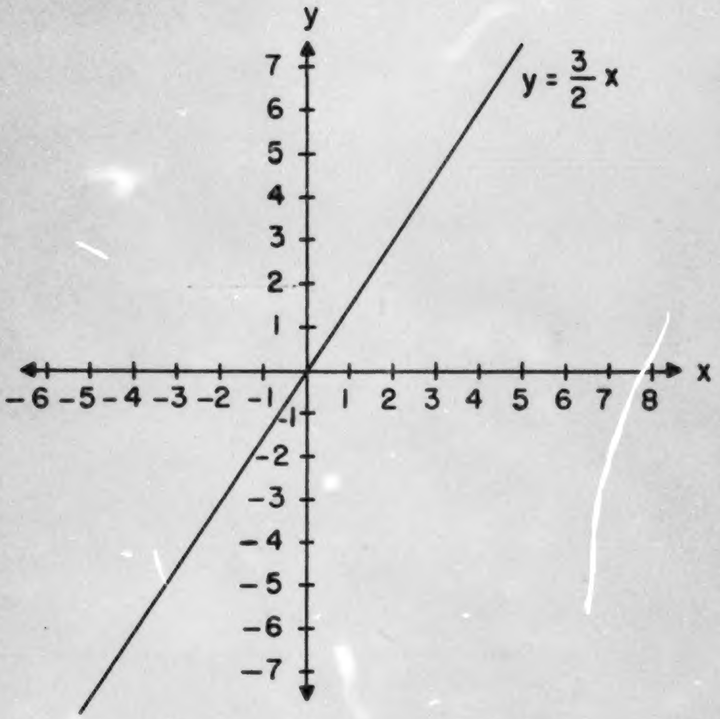
Answers to Exercises 7-1b

1. (1) $k = 32$ or $.32$, (2) $k = 33.9$,
   (3) $k = 7$, (4) $k = 2$, (5) $k = 12$,
   (6) $k = 4$
2. Yes, the equation can be written as $p/s = 4$, but in this form s cannot be zero.
3. $k = 3$; $(y = 3x)$
4. $k = \frac{1}{4}$; $(y = \frac{1}{4}x)$
5. Yes; $20/2 = 50/5 = 70/7 = 100/10 = 110/11$;
   $a/b = 10$ for each pair of corresponding values given;
   $a = 10b$ or $b = \frac{1}{10}a$
6. No; $4/1 \neq 6/2$; a/b is not constant.
   $a = 2b + 2$
7. $d = 40t$
8. $(-4, -6)$; $(-3, -9/2)$; $(-2, -3)$; $(-1, -3/2)$; $(0, 0)$; $(2, 3)$;
   $(5, 15/2)$

9.



10. All the points seem to lie on a line; the curve through the points will be a line.

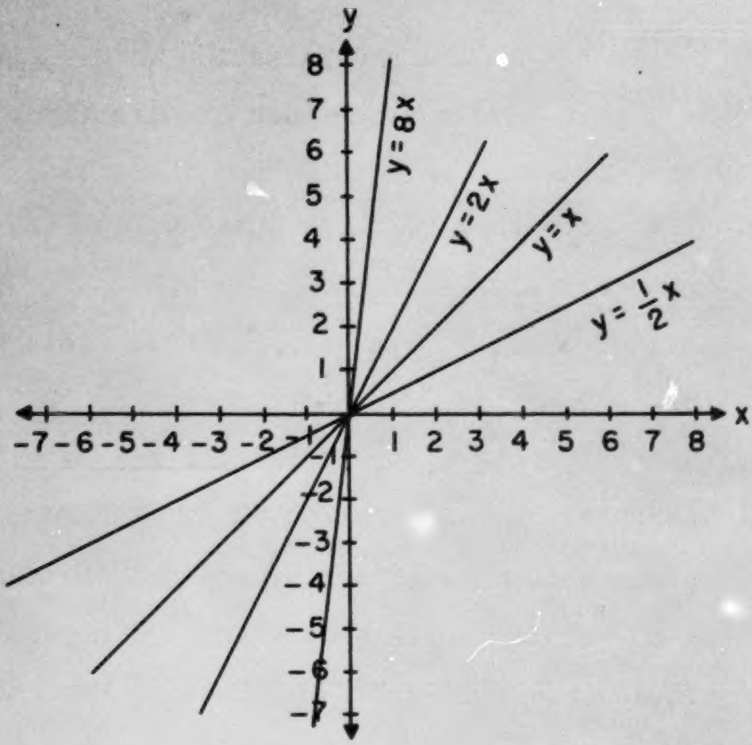


11. Two points are needed. A third point serves as a check.

12. Yes; y is halved; x is multiplied by 10; yes, the statements are true for negative values of x and y.

13. x is halved; y is tripled

14.



15. (0, 0); (0, 0) The constant determines the steepness of the line. The larger the constant, the steeper the line. The smaller the constant, the more gradual the rise.

16. The graphs are lines through the origin. The graph of $y = 100x$ is much steeper than the graph of $y = 5x$.

17. Here are two examples. Many other answers would be correct. The cost, c, of n jet planes: $c = 5{,}000{,}000\,n$. The interest, i dollars, on a principal of p dollars left for one year at 3%: $i = .03p$

18. 2π is the constant.

7-2 Inverse Variation

After the pupils have a clear idea of direct variation, $y = kx$, ask them how else they think two variables, x and y, might vary. Then have them read the introduction and discuss Exercises 7-2a in class.

Bring out in Problem 2 that only positive whole numbers are appropriate for s and n. Thus the domain of n is restricted to counting numbers. Look again at Exercises 7-1a with the pupils and discuss the domain of n in Problem 3. In Exercises 7-2a Problem 3, the distance measure may be any real number. In Problem 4, p may be any whole number and r may be any rational number; in Problem 5, n may be any whole number and r any rational number.

Answers to Exercises 7-2a

1. (a)

Rate	10	20	25	40	50	60	75	80
Time	10	5	4	$2\frac{1}{2}$	2	$1\frac{2}{3}$	$1\frac{1}{3}$	$1\frac{1}{4}$

(b) $rt = 100$

(c) When the rate is doubled, the time is divided by 2.

(d) When t increases, r decreases.

2. (a)

Number of stones in a row	10	12	15	16	30	40
Number of rows	24	20	16	15	8	6

(b) $ns = 240$ (n and s must be counting numbers)

3. (a) When W is 2, D is 18 When D is 1, W is 36
When W is 9, D is 4 When D is 6, W is 6
When W is 18, D is 2 When D is 12, W is 3

(b) As D is doubled, W is halved. As W increases, D decreases.

4. $rp = 200$; When p is doubled, r is divided by 2.

   If the interest rate is doubled, only half as much money on deposit is needed to earn 200 interest.

5. $nl = 20$; When the length of each part is divided by 3, the number of parts is multiplied by 3.

Writing the equations in a column as at the right should help the pupils to see the pattern: in each case the product of two variables is a constant.

$rt = 100$
$ns = 240$
$WD = 36$
$rp = 200$
$nl = 20$

Discuss why k cannot be zero in $zy = k$.

In the exercises, the name of the curve is given merely for information. The kind of variation pictured is important here, not the name of the curve. In some classes it may be advisable to do the graphing with the class as a whole since it is not easy for these young pupils to draw smooth curves. It is helpful to plot as many points as possible to make drawing the curve a little easier.

Answers to Exercises 7-2b

1. $rt = 100$; constant is 100
   $ns = 240$; constant is 240
   $WD = 36$; constant is 36
   $rp = 200$; constant is 200
   $nl = 20$; constant is 20

2. Answers here should include the ideas that in direct variation ($k > 0$), the 2 quantities increase together, preserving a constant ratio; in inverse variation ($k > 0$), as one quantity increases the other decreases in constant ratio. In direct

variation, one quantity is always a given number (k) times the other. In inverse variation, the product of two quantities is a given number (k).

3. $k = 12$; ($xy = 12$)
4. $k = 5$; ($xy = 5$)
5. No; this appears to be an example of direct variation where k is 2. The equation appears to be $b = 2a$.
6. $ab = 12 =$ No; only one pair of corresponding values of a and b is needed.
7. (a) No, since y/x is not constant.

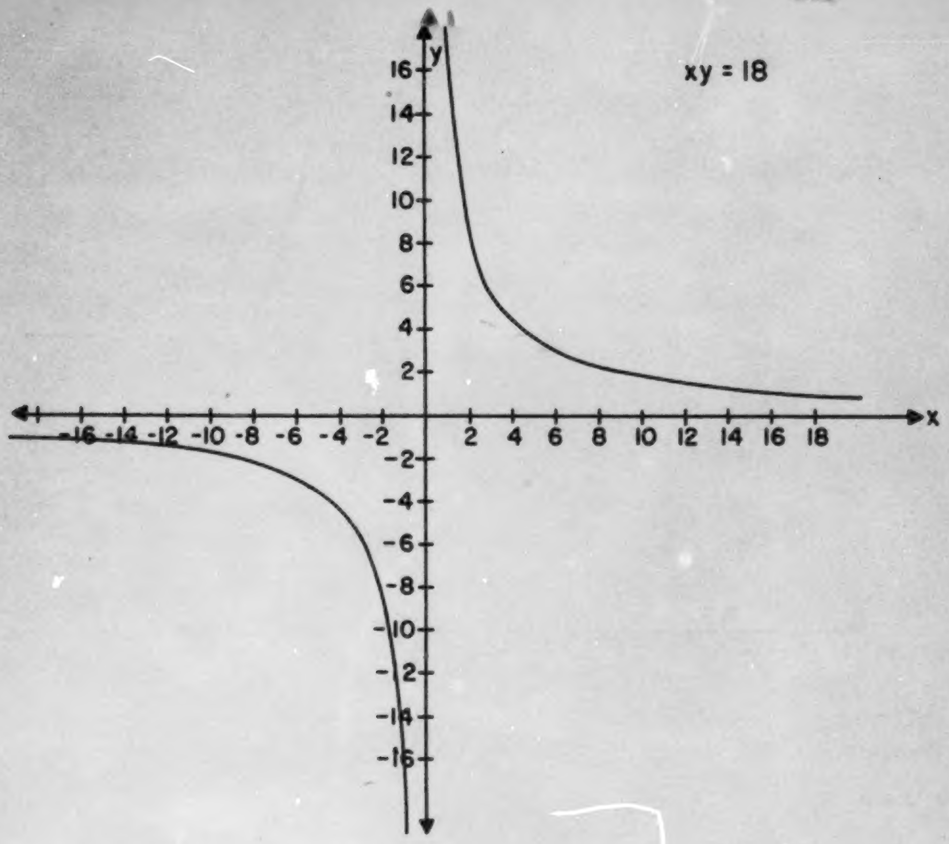
   (b) No, sincy xy is not constant.
8. (a)

x	-4	-3	-2	-1	1	2	3	4	5	6	7	8	9	18
y	$-\frac{9}{2}$	-6	-9	-18	18	9	6	$\frac{9}{2}$	$\frac{18}{5}$	3	$\frac{18}{7}$	$\frac{9}{4}$	2	1

   (b) Neither x nor y may be zero in $xy = 18$.

   The product of zero and any number is zero. There is no number which can be multiplied by zero to get the product 18 .

(c)



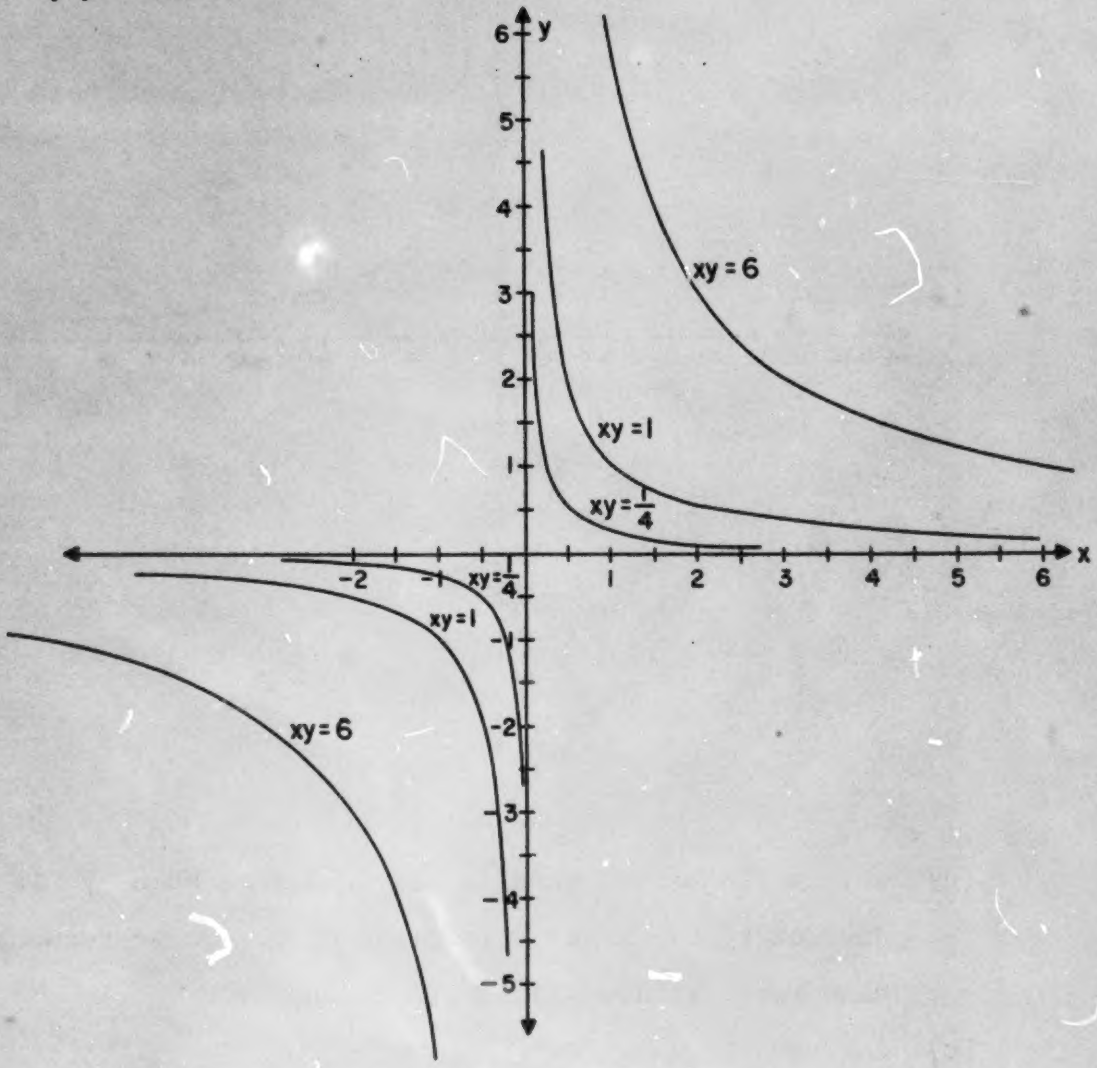
(d) No; y is halved when x is doubled. When y is divided by 6, x is multiplied by 6. These responses hold for negative values of x and y .

(e) inverse

9. (a) y may never be zero in $xy = k$ where $k \neq 0$.

(b) In $xy = k$ if x is doubled, y is divided by 2. If y is multiplied by 8, x is divided by 8 . If x is divided by 10, y is multiplied by 10.

10. (a)



(b) The intersection is the empty set.

(c) The constant effects the way the curve broadens out. For higher values of k the curve is shallow and for smaller values the curve follows the axes into their angle of intersection more closely.

Sometimes pupils are disturbed at the idea that in $xy = k$ the constant k is different in different situations. Help them

see that $xy = k$ covers all cases of inverse variat

see that $xy = k$ covers all cases of inverse variation for positive constants and that in $xy = 12$, for example, the 12 remains constant while x and y vary. Bring out also that $1/x$ is the reciprocal of x and that in the form $y = k \cdot \frac{1}{x}$ we say that y varies directly as the reciprocal of x. This is another way of saying that y varies inversely as x.

Physical interpretations of negative values for the variables in inverse variation frequently depend upon opposites. In $d = rt$, t can be thought of as negative when times before the starting time are considered. Negative values for r may be thought of as meaning motion in the opposite direction. An application in which r is negative is discussed in the overview at the beginning of this commentary.

7-3. Other Types of Variation

There are many types of variation and joint variation, but the only other type discussed in this chapter is the case where one variable varies as the square of another variable. It is advisable to use class discussions extensively as pupils work toward deriving the general statement for themselves. The table of s and A is given below in the answers to Exercises 7-3a.

A class decision should settle the "convenient scale" advised in the problems. This helps pupils clarify the reasons for a particular choice and allows for conformity within the class so that the graphs are more easily checked by the teacher. If a pupil should suggest that the y axis scale be marked at equal intervals: 0, 1, 4, 9, 25, ..., you may wish to tell him that this is often done. It is a logarithmic scale and yields a straight line for the graph of $y = kx^2$. Suggest that for the time being the class will use equal number intervals for equal intervals on the axes in order to see the curve of the graph of $y = kx^2$.

The graph of $y = kx^2$ is called a parabola and, if pupils ask, they should be given the name which otherwise is not emphasized.

In Problem 5 below, question (a) is asked in such a way as to focus attention upon the fact that the constant of variation is not known. The only equation which can be written is $d = kt^2$. The ball falls $9k$ feet in the first 3 seconds but the actual distance is not known.

A discussion of questions in the text preceding the Definition in Section 7-3 follows:

From the graph, the area of a cube 3.5 inches on a side appears to be about 75 square inches.

The length of one side of a cube whose area is 175 square inches appears to be about 5.5 inches.

Exact values cannot be found from the graph.

The area of one face of a cube of side s units is s^2 square units. There are 6 faces. $A = 6s^2$.

When s is 3.5, A is $73\frac{1}{2}$.

If the graph is available, and an approximate answer is acceptable, it is easier to use the graph than the equation. If the graph is not available, but a great many approximate values are wanted, it may be worth-while to draw the graph instead of working out each value from the equation. If exact values are needed, they cannot be found from the graph; the equation should be used.

Answers to Exercises 7-3a

1. (a)

s	1	2	3	4	5	6
s^2	1	4	9	16	25	36
$A = 6s^2$	6	24	54	96	150	216

$\frac{A}{s^2} = 6$ for each pair of corresponding values.

(b) (1) About 30 square inches

(2) About $3\frac{1}{2}$ inches

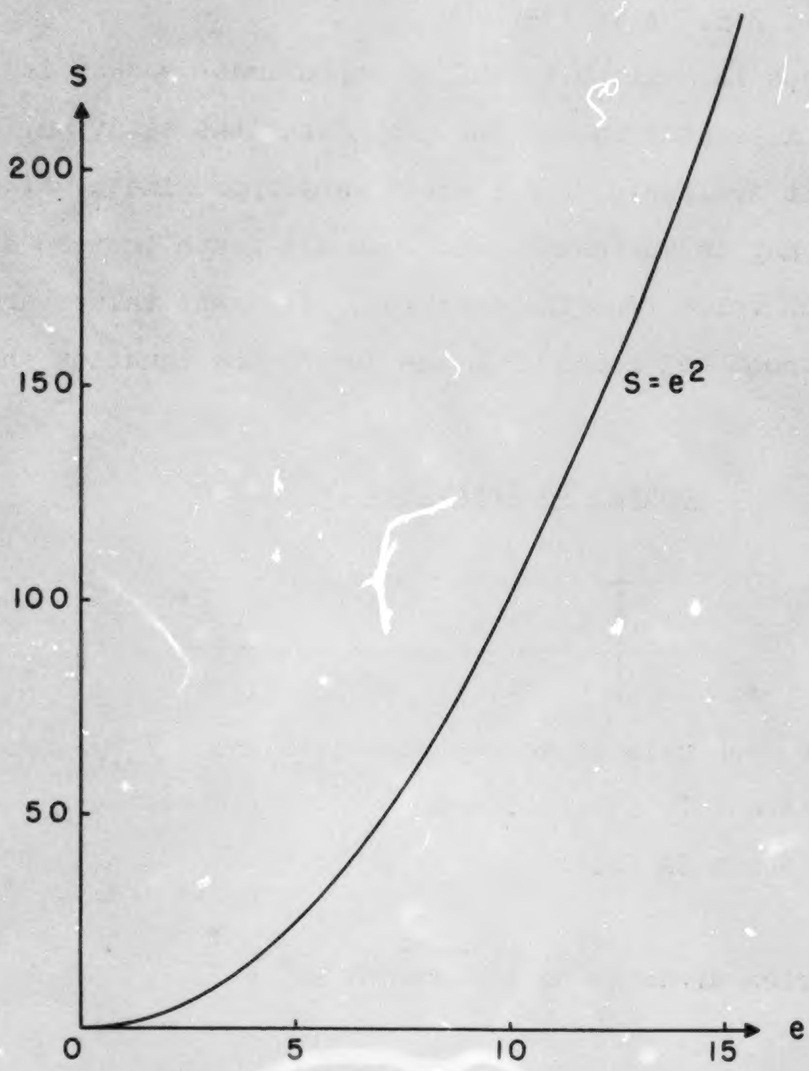
2. (a) $S = e^2$

(b) S varies directly as the square of e .

(c)

e	0	1	2	3	4	5	6	7	8	9
S	0	1	4	9	16	25	36	49	64	81

e	10	11	12	13	14	15
S	100	121	144	169	196	225



(d) (1) 9 sq. cm. (2) 8 cm. (3) 30 sq. cm.

(4) 6.4 cm.

(e) (1) From $s = e^2$, area is 9 sq. cm. (2) 30.25 sq. cm.

3. (a) $E = 4v^2$ (b) $E = 144$ (c) $v = 2$, or $v = -2$

4. (a) $\frac{100}{280}$ or $\frac{5}{14}$ lb.

(b) 25 cents

(c) $C = \frac{1}{4}s^2$

(d) \$10.56

(e) Yes; the cost for a plot 75 ft. on a side is \$14.06.

5. (a) $d = kt^2$

(b) No; it falls 9k feet, but we do not know k.

(c) $d = 16t^2$

(d) 400 ft.

Be sure pupils understand the general character of the equation $y = kx^n$. The exercises ask for a number of graphs which you may prefer to have the pupils do in class as a group activity. Perhaps in Problem 2 you may wish to separate the class into groups of 3 pupils with each pupil drawing a different one of the 3 graphs. As the 3 pupils compare their graphs (drawn on the same scale) they will be able to answer the questions. A similar procedure may achieve the ends sought but with less drudgery in Problem 5.

Answers to Exercises 7-3b

1. (a)

x	-6	-4	-2	0	2	4	6
$y = \frac{1}{2}x^2$	18	8	2	0	2	8	18

(b)

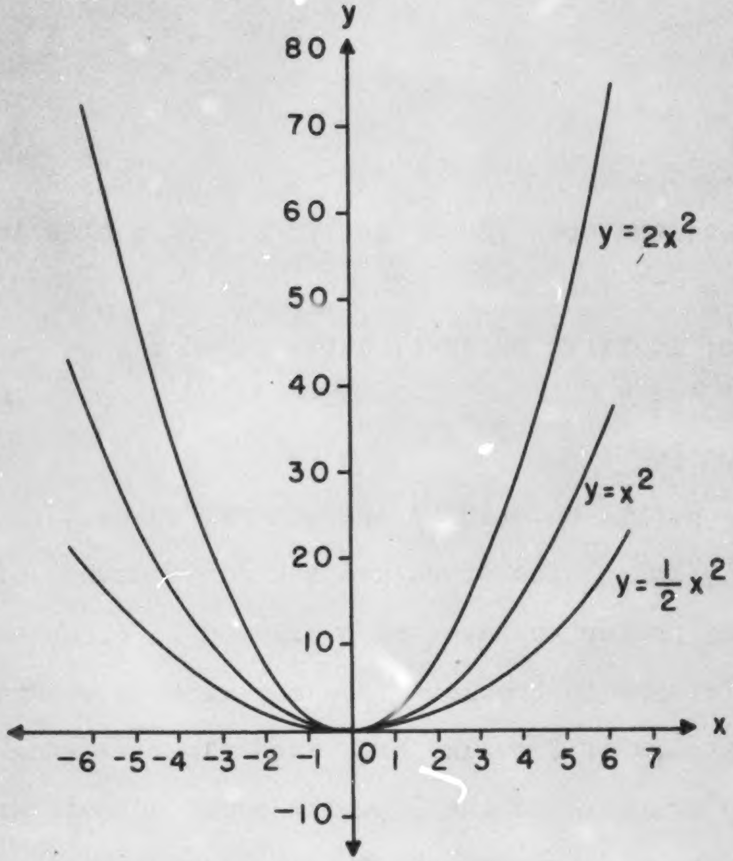
x	-6	-4	-2	0	2	4	6
$y = x^2$	36	16	4	0	4	16	36

(c)

x	-6	-4	-2	0	2	4	6
$y = 2x^2$	72	32	8	0	8	32	72

(d) directly as x^2

2.



(a) If the value of k is doubled, the graph of $y = kx^2$ appears to curve upward more rapidly than formerly. Each y is twice the former value.

(b) If the value of k is multiplied by 4, the graph of $y = kx^2$ appears to curve upward still more rapidly. Each y is 4 times the former value.

(c) Changing k changes the ratio of y to x^2.

When k is increased, each y value is increased in direct proportion to the increase in k.

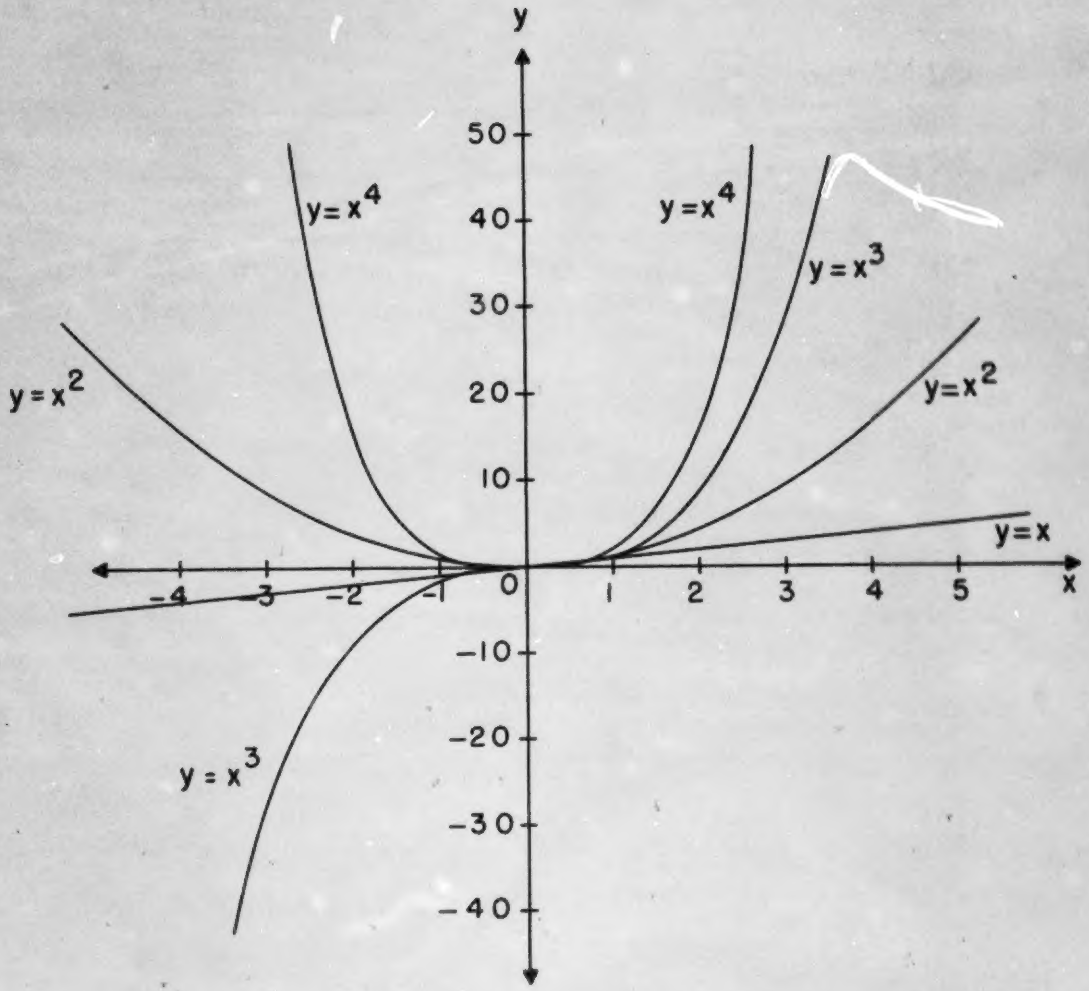
For large values of k, the curve is steeper and rises faster than for small values of k.

3. If x is doubled, y is multiplied by 4.

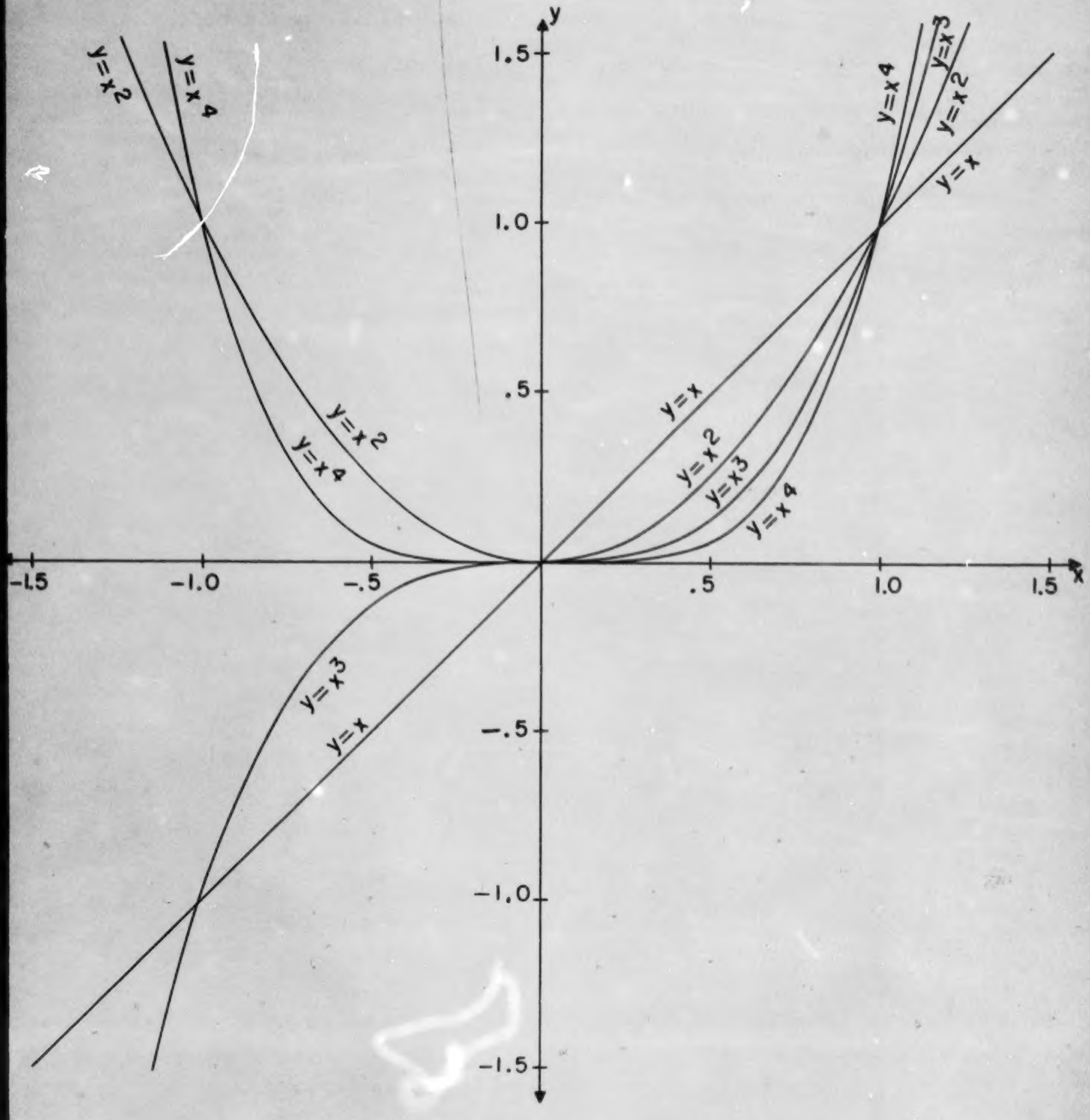
4.

x	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
(a) $y = x$	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
(b) $y = x^2$	9	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4	9
(c) $y = x^3$	-27	-8	-1	$-\frac{1}{8}$	0	$\frac{1}{8}$	1	8	27
(d) $y = x^4$	81	16	1	$\frac{1}{16}$	0	$\frac{1}{16}$	1	16	81

5. (a)


y
x
50
40
30
20
10
0
-10
-20
-30
-40
-4
-3
-2
-1
1
2
3
4
5
$y=x^4$
$y=x^4$
$y=x^3$
$y=x^3$
$y=x^2$
$y=x^2$
$y=x$

A portion of the graphs on the previous page has been re-drawn here using a different scale to show the relative positions of the curves for values of x between -1 and 1.



(a) As n is increased, the graph of $y = x^n$ becomes steeper more rapidly. The values of y increase at a rapid rate for even small integral increases for x . If n is even, y is never negative; if n is odd, y is negative for negative values of x .

(b) If the same change is made in n as in k (doubling, for example) the change in n has the greater effect upon the graph of $y = kx^n$.

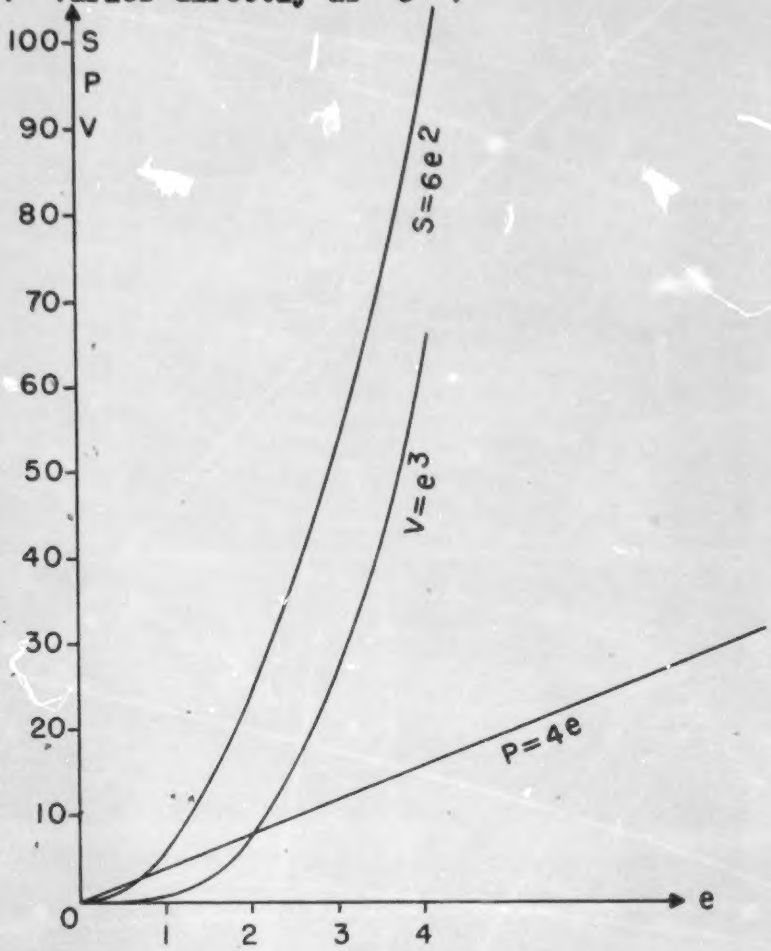
6. (a) $P = 4e$; $S = 6e^2$; $V = e^3$

(b) (1) P varies directly as e .

(2) S varies directly as e^2 .

V varies directly as e^3 .

(c)



(d) From the graph: $P = 10$; $S = 37\frac{1}{2}$; $V = 16$

From the equation: $P = 10$; $S = 37\frac{1}{2}$; $V = 15\frac{5}{8}$

(e) V will be the largest when $e = 10$ cm. and P will be smallest. $P = 40$; $S = 600$; $V = 1000$

7-4. Summary and Review

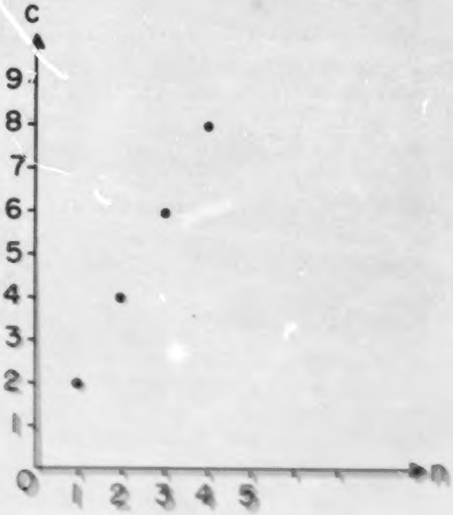
The summary collects and organizes the information presented in the chapter for easy reference. The problems review the three kinds of variation and provide for the use of the ideas in simple applications. Some additional sample questions are included here after the answers to Exercises 7-4

Answers to Exercises 7-4

1. 40
2. 32/5
3. 100
4. $\frac{A_1}{A_2} = \frac{2^2}{3^2} = \frac{4}{9}$
5. (a) 28 pounds
   (b) 7 inches
6. 112 lb. per sq. in.
7. No
8. "xy = k", not "x + y = k", is the equation which shows inverse variation
9. $5\frac{11}{68}$ inches; direct variation
10. In direct variation, $y = kx$ not $y = k + x$.
11. 11[illegible] inches
12. inverse variation
13. direct variation; [illegible]
14. V is multiplied by 8; the constant is [illegible] .

15. (0, 0); (1, 2); (2, 4); (3, 6); (4, 8);

The graph will consist of separate points as shown.



Sample Test Questions

1. (a) What kind of variation is indicated in $C = \pi d$?

Ans.: Direct variation.

(b) What is the constant of proportionality? Ans.: π

2. (a) What kind of variation is indicated in $A = \pi r^2$?

Ans.: Variation as the square.

(b) What is the constant of proportionality? Ans.: π

(c) What happens to the area within a circle when the radius is multiplied by 10?

Ans.: The area is multiplied by 100.

3. (a) If a class must pay 30 to charter a motor coach for a field trip, how does the cost per pupil depend upon the number of pupils who take the trip?

Ans.: The cost per pupil varies inversely as the number of pupils.

(b) Use n for the number of pupils who go, and c for the cost per pupil and write an equation connecting n and c. Ans.: $nc = 30$

(c) What kind of variation is indicated?

Ans.: Inverse variation.

4. (a) When a platform is moved on rollers, the number of inches the platform goes forward (p) is 2 times the number of inches the rollers go (r). Write an equation connecting p and r.

Ans: $p = 2r$.

(b) What kind of variation is represented?

Ans.: Direct variation.

5. (a) Four men can do in 10 days what 8 men can do in 5 days. This is an example of what kind of variation?

Ans.: Inverse variation.

(b) How long should it take 2 men to do the same work?

Ans.: 20 days.

UNIT 8

NON-METRIC POLYHEDRONS

General Remarks

The general purposes of this unit are similar to those for Chapter 4 of Volume I. It is suggested that teachers read the beginning of the Commentary for Teachers of that chapter. In it there is more emphasis on the use of models in understanding spatial geometry. There is correspondingly less emphasis on "sets" as such. This unit is more geometry than "sets". However, the terminology of sets, intersections, and unions is used throughout this unit. The student should already have some familiarity with these ideas. A quick run over of the ideas of Sections 4-1 to 4-8 of Chapter 4 of Volume I would be useful as background for most students. However, if sets, intersections, unions, and some geometrical terms can be explained as you go along, this material should be teachable without too much specific background on the part of the student.

The use of terms like "simplex" and the distinctions between 2- and 3-dimensional polyhedrons should contribute to precision of thought and language.

Reading. The text is written with the intention that the pupils read (and study) it. There may be occasional passages which some pupils will not follow easily. They should be encouraged to read beyond these passages (if there are any).

Materials. The students should be encouraged to use cardboard or oak tag for their models of tetrahedrons and cubes. They will be asked to draw on the surfaces of these later. It is suggested that each student contribute one model of a regular tetrahedron and one of a cube for later class use. These will be needed by the teacher later in the study of polyhedrons. Each student should keep a model of a cube and a model of a tetrahedron at home for use in homework. In this commentary for Section 8-4 and 8-7 there are several suggestions for making models from blocks of wood. The boys in the class should be encouraged to do this.

Motivation. Why put material like this in the 8th grade curriculum? There seem to be several good reasons for it.

1. It helps develop spatial intuition and understanding.
2. It emphasizes in another context the role of mathematics in reducing things to their simplest elements.
3. It affords other ways of looking at objects in the world about us and raises fundamental questions about these.
4. It illustrates types of mathematical (geometric) reasoning and approaches to problems.
5. It gives an interesting insight into the meaning of dimension.

For the year 1959-60 this material should be regarded as experimental. It has not been taught traditionally.

Time. The material of this unit is recommended for study for a period of about two weeks. If class interest is high, some extra time could profitably be used.

A Word of Warning. In preparing themselves on this material, teachers should not expect to get it at one reading. It will be easier if they assimilate the terminology and points of view gradually.

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8-1. Tetrahedrons

It is probably desirable for the teacher to follow the instructions in the text to make in class either a model of a regular tetrahedron or a model of a non-regular tetrahedron or both.

Have each student leave one model of a regular tetrahedron with you. Have them keep one model of a non-regular tetrahedron to use at home.

Some boys might like to make models of tetrahedrons by sawing wood blocks.

Answers to Questions in Text

The edges of (PQRS) and (PQ), (PR), (PS), (QR), (QS), and (RS).
The faces of (PQRS) are (PQR), (PQS), (PRS), and (QRS).

Answers to Exercises 8-1

1. (Construction)
2. (Construction)
3. The measure of the new angle DAC must not be more than the sum of the measures of angles BAC and BAD nor less than their differences. Otherwise, it won't fit. (This could be tried in class.)

8-2. Simplexes

In Chapter 4 we regarded points, lines, and planes as our "building blocks". Here (with some prior notion of lines and planes) we will use points, segments, triangular regions, and solid tetrahedrons as our "building blocks".

The discussion about "taking points between" and "dimension" probably needs to be read more than once by pupils. After one gets the general idea the material should be able to be read. A class discussion of the idea prior to the pupils reading the text may be a good idea.

The text concerning "betweenness" illustrates an important fact about reading some mathematics. One sometimes has an easier time reading for general ideas and then rereading to fill in details.

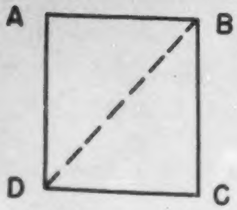
Answers to Exercises 8-2

1. (a) 3
   (b) 3
2. 2
3. Better students might be encouraged to try to draw
4. figures for each of these problems.
5. The vertices of a 2-simplex are blue.
   The edges will be red.
   The interior will be green.
   No. Just two.
6. The vertices of the model are blue.
   The edges will be red.
   The faces will be green.
   The solid tetrahedron.

Note: When we say "color points" what we mean, of course, is "color the representations of the points on the paper or model".

Answers to Exercises 8-3

1. Constructions
2. There are a number of possible answers... (PABD) and (PBCD) could be two of the eight if one face is labeled thusly:



3. 24 2-simplexes.

4. 24

Yes, you can make the comparison.

The top vertices of the six pyramids would correspond to the six points in the middle of the faces of the cube.

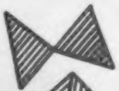
8-4. Polyhedrons

In class it would be a good idea to hold or fasten models of tetrahedrons together to indicate various 3-dimensional polyhedrons. Also, blocks of wood with some corners and edges sawed off (and maybe with a wedge-shaped piece removed) are good models of 3-dimensional polyhedrons.

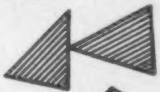
Models of tetrahedrons should be held together to indicate how the intersection of two could be exactly an edge of one but not an edge of the other. (They would not intersect nicely.) Other examples can also be shown here.

Answers to Exercises 8-4

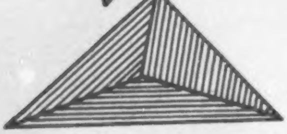
1. (a)



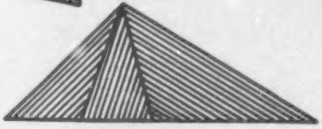
(b)



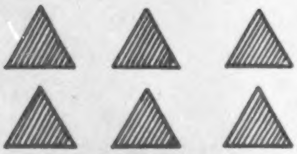
2.



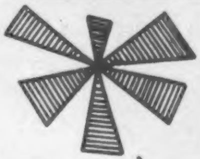
or



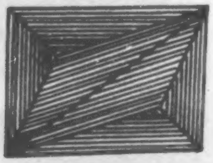
3. (a)



(b)



(c)

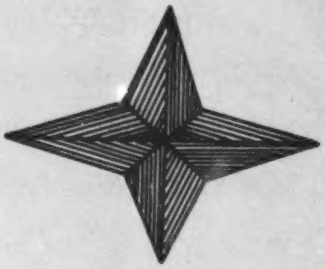


(one possible answer)

4. Draw the segments (BD), (GE), and (HE).

5. (a) Draw segments (BD), (BH), (DF), (FH), and either (BF) or (HD).

   (b)



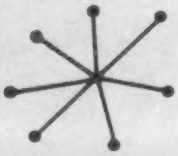
6. (a) (PQR), (XYZ), (XPQ), (XYQ), (XZQ), (ZQR), (XZP), and (PRZ).
   (b) (FPQR), (FXYZ), (FXPQ), (FXYQ), (FXZQ), (FZQR), (FXZP), and (FPRZ).

8-5. One-Dimensional Polyhedrons

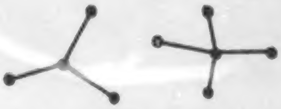
The pupils may enjoy drawing all sorts of simple closed polygons. Some should be in a plane. But they also should be encouraged to do some on surfaces of cubes, tetrahedrons, wooden blocks, etc.

Answers to Exercises 8-5

1. 2 paths from A to B.
   3 simple closed polygons.
2. (a) 7
   (b) PQSP, PRSP, PQRP, QRSQ, PQRSP, PRQSP, PRSQP.
   (c) PQRSP (We are naming the vertices in order returning to a starting point.)
3. PABCEDFQ, PCBAFEDQ, PEDCBAFQ (there are others)
4.



or



or such.

5. There are many ways of doing it.

8-6. Two-Dimensional Polyhedrons

This section gives a good opportunity to draw a lot of 2-dimensional simplexes and polyhedrons and to illustrate again sets of 2-simplexes which intersect nicely.

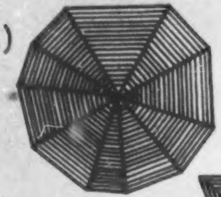
Some drawing and illustrations should be in the plane and some should be on surfaces or 3-dimensional polyhedrons.

Answers to Exercises 8-6

1.

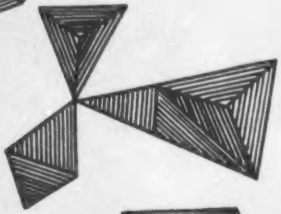
	F	E	V
Tetrahedon	4	6	4
Cube	6	12	8
Cube (with simplexes)	12	18	8

2. (a)



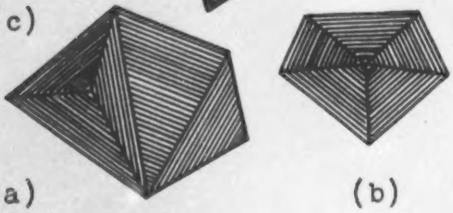
many other possibilities.

(b)



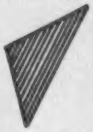
many other possibilities.

(c)

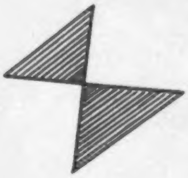


many other possibilities.

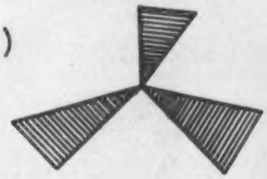
3. (a)



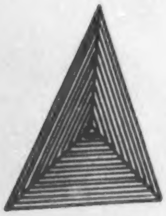
(b)



(c)



4.



No, there won't be four. The union of the first three must itself be 2-simplex.

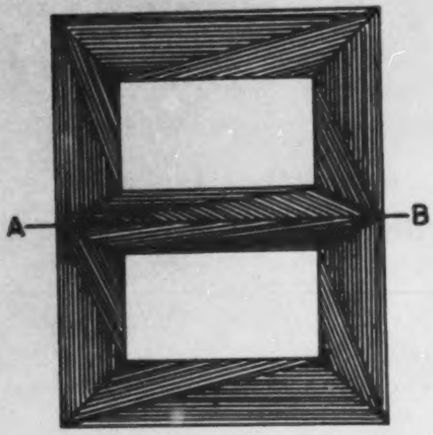
5. F = 24 E = 36 V = 14.

6. 6 sets of 4-simplexes each.

12 original edges.

6 sets of 4 new edges each.

7.



This can be done in many ways. We have first subdivided it along AB into two figures like the second in the text of this section.

8-7. Three-Dimensional Polyhedrons

In this section and the next section there are many fine opportunities to use the models which the pupils have prepared. It is suggested that the class be split into three or four parts and that each part take several models of regular 3" tetrahedra and fasten them together (with cellulose tape or paste) to produce some rather peculiar looking 3-dimensional polyhedrons. They can also fasten models of cubes together to produce odd looking polyhedrons.

There is a good opportunity here to encourage some of the boys in the class to make models of 3-dimensional polyhedrons out of blocks of wood. Start with a block of wood and saw corners or edges off of it and possibly notches out of it. The surface you get will be a simple surface as long as no holes are punched through the solid. You could not punch holes through with an ordinary saw. Be sure that the surface is made up only of flat portions. A surface like this could be covered with paper and then colored or drawn on. It could then be recovered with paper for further use. It is interesting that no matter how the block is cut up (without holes) every simple closed polygon on it will separate the surface into just two pieces. Also the models can be used for examples of the Euler Formula in the next section. The faces, edges, and vertices can be counted rather easily on objects like this.

Make a model of a "square doughnut" out of 8 cubes as suggested in the text. The surface will not be simple.

Answers to Exercises 8-7

The exercises are all construction problems. The teacher should assign other problems as appropriate to the class.

8-8. Counting Vertices, Edges, and Faces--The Euler Formula

Use models of tetrahedrons and cubes in explaining the material of this section to the class.

The counting of faces, edges, and vertices on any simple surface should be interesting and informative for the pupils. Use models made of wood for some of the examples.

Use the model of the "square doughnut" of the last section to help pupils count faces on this type of surface.

The teacher should assign particular problems on counting faces, edges, and vertices as appropriate for the particular class.

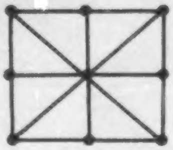
The teacher should subdivide the surface of a model of a cube in an irregular way and count faces, vertices, and edges as indicated in the text.

Answers to Exercises 8-8

1. The count is 12 faces, 18 edges, and 8 vertices.

   The 2-simplexes of the subdivision of the tetrahedron can be corresponded to the faces of the solid which is the union of the five tetrahedron.

2. A face looks like

   26 vertices.

   48 faces.

   72 edges (48 on interiors of faces of cube and 24 on edges of cube).

3. F + V - E should be 2.

UNIT 9

VOLUMES AND SURFACE AREAS

General Remarks

The development of the main portions of this unit is along empirical and intutitive lines. We have made generous provisions for discovery activities. The entire unit is built around the construction and frequent use of the various models the pupils are to study.

At the end of this unit you will find diagrams for constructing models of the geometric solids we will consider. We suggest you give the following instructions of steps to be taken in the construction of the models:

(a) Place a sheet of stiff paper such as oak tag behind the diagram in the text.

(b) With a compass point or pin point perforate the text page at all the verticles.

(c) Remove the stiff paper and join the holes by using a sharp pencil and a straight edge so that your diagram resembles that of the text.

(d) With a pair of scissors cut along the boundary lines of the figure or the stiff paper.

(e) Fold your cut out along the inner edges of your solid.

(f) Use glue to paste the tabs on the inner edges of your solid.

(g) Seal each such edge by using scotch tape (they are to be used as containers).

(h) Take good care of your models since they are to be used

(h) Take good care of your models since they are to be used frequently.

We suggest that you take some time in class helping the class construct the first models. You may prefer that the pupils work independently or in groups. Be sure that they construct their models carefully. You will notice that one of the bases is not permanently sealed since this model will be used as a container. It is convenient at times to temporarily attach this base also.

We have assumed that your pupils are somewhat familiar with the following geometric terms: point, line, line segment, ray, parallel lines, intersecting lines, perpendicular lines, angles, measure of the interior of an angle, triangle, altitude and base of a triangle, base and altitude of a rectangle or length and width of a rectangle, rhombus, square, regular polygon, hexagon, perimeter of a polygon, circumference of a circle, Pythagorean property and congruent triangles.

You might perhaps take an inventory to see whether or not this is true. If preliminary work needs to be done we highly recommend chapters 7 and 9 of SMSG '59-'60 text for Junior High School Mathematics, Volume I. The subject of measurement is extensively developed in chapter 7. The subject of parallels, parallelogram, triangles, and right prisms are covered in chapter 9.

If the make-up of your class is such that the pupil already knows a good deal about areas of plane figures and volumes of right prisms respectively, sections one and three could serve as a brief review of the main ideas.

If on the other hand the make-up of your class is such that

If on the other hand the make-up of your class is such that your pupils have had little or no preparation for area and volume, sections one and three should definitely be expanded upon. In these sections we have done more "telling" than we would ordinarily do if we were introducing the ideas of area and volume for the first time. Chapters 7 and 9 of SMSG '59-'60 Junior High School Mathematics, Volume I, text should most certainly be consulted in expanding upon the brief treatment in sections 1 and 3 of this unit.

Encourage your pupils to estimate areas and volumes prior to calculations and experiments. We feel that these activities will help them to better appreciate the ideas of area and volume than the results obtained by rote calculations alone. It will be necessary for you to supply such problems as they have not been included in the text.

The activities of finding volumes by fitting cubic inches and fractions of cubic inches which pupils have constructed and of pouring sand and salt into the interiors of the various prisms should also help them to understand the concept of volume.

You will notice that we have not done much in the way of articulating with the ideas developed in Unit 8. We suggest you take advantage of these ideas whenever they will help you to clarify and unify the basic ideas presented in this Unit.

The dimensions were given for the various patterns for your convenience.

The surfaces of each of these tetrahedrons may be separated into its triangles and flattened out on a plane as in the following patterns in Figure 4. The first of these of course is the given pattern for the original tetrahedron, though now drawn here to the same scale.

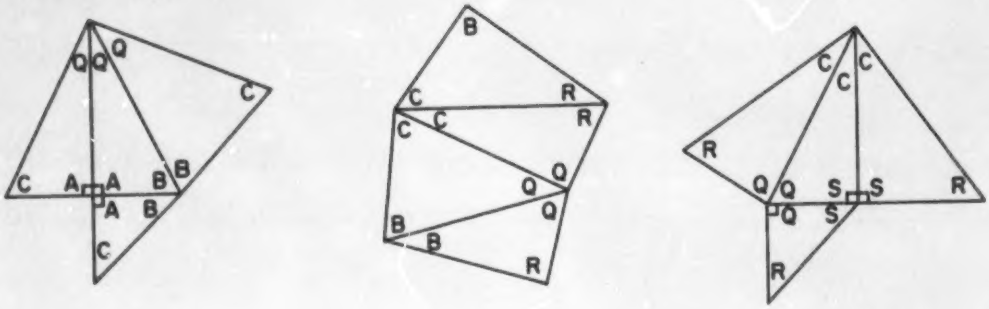


Figure 4

The question is whether we have enough information given us in Figure 4a to allow us to draw Figures 4b and 4c so we can construct the required models from them. Notice that from Figure 4a, which was given us, we know segments $\overline{AB}$, $\overline{AC}$, $\overline{BC}$, $\overline{AQ}$, $\overline{BQ}$, $\overline{CQ}$. Moreover, by looking at Figure 2 we see that the following sets of segments have equal lengths:

$\overline{PR}$, $\overline{CS}$, $\overline{AQ}$	(lateral edges of a prism)
$\overline{AB}$, $\overline{QR}$	(opposide sides of a rectangle)
$\overline{BC}$, $\overline{RS}$	"
$\overline{AC}$, $\overline{QS}$	"

Thus all segments just named are know. Recall that it is easy to draw a triangle by making use of a compass if the three sides have known lengths.

Now look at Figure 4c. Every segment in it except CR is in the above list and so is known. Thus we can easily draw triangle QRS and triangle QSC. Moreover, we know angle CSR is a right angle since in Figure 2 CS is perpendicular to the plane of QRS. Thus, to finish Figure 4b, simply extend QS to the right a distance equal to the length of SR and draw CR . With CR known the remaining triangle can easily be drawn since all sides are now known.

Figure 4b can now be easily drawn since all segments are known, CR having been just found in drawing Figure 4c. Thus Figures 4b and 4c can actually be drawn, and then folded sogether to produce the desired figures.

It may be worth noting that this method always works as long as AQ is perpendicular to the plane of ABC, but breaks down when this is not true as, for example, in our Model 17. If attempted, everything goes successfully until you reach the point, in drawing Figure 4c, or drawing triangle CSR . In this case angle CSR is not a right angle and you do not know in what direction to draw SR . Hence you do not know how to finish the figure. Of course there is a correct angle CSR , but the student currently has no means of finding it.

Answer Key

Unit 9

Section 9-1a

1. (a) 48 square inches
   (b) 1/3 square feet
   (c) 169 square inches
   (d) 1 25/144 square feet
   (e) 72 square inches
   (f) 1/2 square feet

2. (a) 10 square inches
   (b) 70 square inches
   (c) 20 square inches

Section 9-1b

1. (a) $17\frac{1}{2}$ square inches
   (b) 225 square feet
   (c) $64\sqrt{3}$ square feet
   (d) $16\sqrt{3}$ square yards
   (e) 1,038 square inches
   (f) 2,750 square inches
   (g) 17,400 square feet
   (h) 3,080 square inches

Section 9-1c

1. (a) 25π square inches
   (b) 100π square inches
   (c) 400π square inches
   (d) 20.25π square inches

(e) 100π square inches

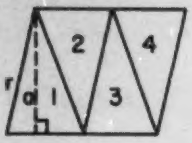
(f) 225π square inches

2. Constructions

3. 25π, 100π, 400π; ∴ doubling the radius quadruples the area

4. (a) **BRAINBUSTER:**

Arrange the 20 congruent triangles as in the following figure



etc.

$a \approx r, \quad b \approx \frac{1}{2}c$

.. $bh = (r)(\pi r) = \pi r^2$

(b) We would say that the area of the interior of the circle is approximately equal to the area of the interior of the circumscribed polygon. It will always be less than the area of the circumscribed polygon since there will always be vertices of the polygon which are not points of the circle. Therefore there is always some portion of the area of the interior of the circumscribed polygon which is not contained in the area of the interior of the circle. However for larger values of n the areas are very close and we can think of the area of the interior of the circle as the lower limit of the area of snterior of the circumscribed polygon.

Thus:

(Area of interior of inscribed polygon)<(Area of interior of circle)<(Area of interior of circumscribed polygon). For very large values of n we can replace (<) by (≈).

Exercises 9-2

1.

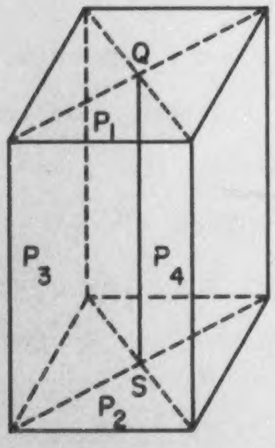
2. Constructions

3. Constructions

4. l_1 and l_2 cannot be skew lines since they lie in the same plane r . l_1 and l_2 cannot intersect since they lie in parallel planes p_1 and p_2 . Hence l_1 and l_2 can only be parallel.

5. (a) A line perpendicular to a plane is perpendicular to all lines in that plane.
   (b) The side opposite the right angle of a right triangle is the hypotenuse of the right triangle.
   (c) The hypotenuse of a right triangle is the longest side of a right triangle.

6.



(a) Draw P_3 and P_4 containing $\overline{QS}$.

(b) P_3 and P_4 are perpendicular to P_2 since they contain $\overline{QS}$ a line perpendicular to P_2 .

(c) P_3 and P_4 are perpendicular to P_1 since P_1 is parallel to P_2 .

(d) $\overline{QS}$ is perpendicular to P_1 since P_3 and P_4 contain $\overline{QS}$.

Exercises 9-3a

1. Construction
2. Counting of cubic inches and fraction thereof.
3. (a) 5 cubic inches
   (b) 10 cubic inches
   (c) $4\frac{1}{2}$ cubic inches
4. They are the same.
5. (a) 16 square inches
   (b) 28 square inches
   (c) $16\frac{1}{2}$ square inches
6. Constructions
7. Constructions

Exercises 9-3b

1. (a) Perimeter of base of model 4 is 6 inches.
   Perimeter of base of model 5 is 6 inches.
   Perimeter of base of model 7 is 6 inches.
   Perimeter of base of model 8 is 6 inches.
   (b) Yes.
   (c) Rhombus based right prism, $V = 5 \cdot 2$ cubic inches
   Right rectangular prism, $V = 10$ cubic inches
   Hexagon based right prism, $V = 10 \cdot 4$ cubic inches
   Right circular cylinder, $V = 11 \cdot 5$ cubic inches
   (d) No.
   (e) See (c)
   (f) The interior of a circle has the maximum area.

2. Volume for model 4, V = 10 cubic inches

Volume for model 6, V = 10 cubic inches

Yes. They are equal.

Perimeter of base of model 4; P = 6 inches

Perimeter of base of model 6; P = 7 1/4 inches

No. The perimeters are not equal.

Exercises 9-4

1. Class activity
2. Constructions
3. Constructions
4. Yes. It is the perpendicular distance between the bases.
5. No. It is not the perpendicular distance between the bases.
6. Too large, since the lateral edge of an oblique prism is greater than the height.

Exercises 9-5

1. (a-c) Draw line segments $\overline{QS}$ and $\overline{QT}$. Consider triangles AQS and AQT to be right triangles since AQ is perpendicular to the base and every line in the interior of the base. Now AS and AT are hypotenuses of their respective triangles. $\therefore$ AS and AT are 5 inches in length (3, 4, 5 right triangles)

   (d) Yes.

   (e) Yes.

2. (a) It would work for any regular polygon as a base.

   (b) It would apply to any lengths.

   (c) Lateral edges

3. (a) Use the Pythagorean property.

(b) $(AS)^2 = (AO)^2 + (SQ)^2$

$(13)^2 = 12^2 + (SQ)^2$

$169 - 144 = (SQ)^2$

$25 = (SQ)^2$

$5 = (SQ)$

same development for QT

$5 = QT$

(c) Yes.

(d) Yes.

(e) Yes.

(f) Yes.

4. (a) No.

   (b) Lateral, a regular.

5. Construction

6. One.

7. 600 square inches.

8. (a) 340 square inches.

   (b) 13 inches.

9. (a) 360 square feet

   (b) $\sqrt{194}$

Exercise 9-6

1. 3 cubic inches.
2. Check (class activity)
3. 3 cubic inches. Yes.
4. 6 equilateral triangles each of whose interiors measure $\sqrt{3/4}$ square inches $\therefore$ $6/4\sqrt{3}$ square inches or $1{\cdot}5\sqrt{3}$ square inches.
5. Check (class activities)
6. 81 square feet, 9 feet.

7. Construction

Exercises 9-7

1. $T = \pi r^2 + \pi rs$ or $T \div \pi r\ (r+s)$

2. $L = 36\pi$ sq. ft., $T = 45\pi$ sq. ft.

3. $r = 9$ feet, $L = 135\pi$ sq. ft., $T = 216\pi$ sq. ft., $V = 324\pi$ cu. ft.

4. $h = 3$ feet, $s + \sqrt{13}$ feet, $L = 2\pi\ \sqrt{13}$ sq. ft.

5. Construction

BRAINBUSTER: This is an extremely difficult problem for students, and few if any will make any headway with it. In case it creates interest however, here is a discussion analyzing the problem and giving the solution.

If the model is assembled, it will look essentially like Figure 1 below. Note that AQ is perpendicular to the plane of triangle ABC.

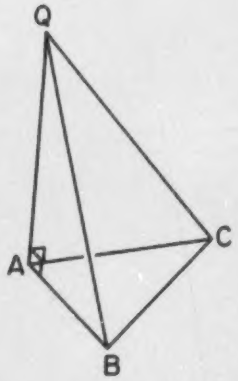


Figure 1

UNIT 10

RELATIVE ERROR

General Remarks

The process of measurement plays such an important part in contemporary life that everyone should have a clear understanding of its nature. A substantial part of the arithmetic taught in the elementary school relates to measurement. Most of the early work in measurement is designed to familiarize children with common units of measure and their use, and with the ratios between them.

The basic concept to be developed in this unit is that the process of measurement of a single thing yields a number which represents the approximate number of units. This is in contrast to the process of counting separate objects, which yields an exact number. When the number of separate objects is rounded or estimated, the resulting number is treated as an approximation in the same sense as a measurement. Since measurements are approximate, calculations made with measurements, such as sums or products, yield results which are also approximate.

It is a fundamental principle of scientific discipline that no direct measurement can be achieved with complete and total accuracy. (Accuracy here is being used in the sense of being correct, not in the technical sense that is developed later in the chapter. The teacher here might note that "accurate" as a word with a definite meaning and "correct" in the usual sense might be quite different.) Many errors enter into the act of measurement. They are:

(a) Mistakes
(b) Instrumental errors
(c) External errors
(d) Personal errors

Preventable mistakes come about through ignorance, carelessness, or improper use of the measuring instrument. These mistakes can be corrected by means of a system of checks, by proper education and by careful inspection during the measuring process.

Instrumental errors are inherent to the measuring device. These errors are the results of imperfections in materials and in the manufacturing process. They are also the end products of economics in the system of manufacturing. To make an instrument more precise requires better materials, more time and superior manufacturing techniques, all of which cause higher costs.

External errors are environmental errors introduced by causes known and unknown. Wind currents may affect the accurate weighing of an object. Temperature changes may cause corresponding changes in the dimensions of both objects measured and the measuring instrument. Some of these errors can be controlled to a degree by measuring in a controlled environment. Some of the errors of known origin can be corrected by applying correction factors, which, in effect, nullify the external errors. Some external errors are so complicated in origin that they defy analysis and complete correction.

Personal errors are due to human imperfections. Most measurements have human beings involved in one way or another. No two persons react to a situation in exactly the same manner. Nevertheless, personal errors can be reduced by proper selection and training of personnel in the measuring operation. (This latter factor may explain to the teachers why industries spend great sums of money simply teaching people how to read measuring instruments. Pupils may then see the consequent importance of this in school work.)

Errors are also characterized as determinate and indeterminate. We have already discussed some of the consequences of these errors under types of errors above. Determinate errors are ones that occur in a known pattern throughout a series of measurements. Such errors can be analyzed, accounted for and corrected. Instrumental errors and external errors caused by changes in temperature are of the determinate variety.

Indeterminate errors are the more insidious and more difficult to recognize and cope with. They have no modus operandi and their occurrence is haphazard and inconsistent. Human errors introduced by fatigue and anxiety are of the indeterminate variety. A sudden gust of wind during the measuring operation will introduce an indeterminate error. The only way to discover and manage this type of error is by repeated observations and measurements.

The unit selected for a given measurement should be suitable for the thing to be measured, and for the purpose for which the measurement is to be used. The unit is not necessarily a standard unit, but may be a multiple of a standard unit, or a fraction of a standard unit. For example, the height of an airplane above the ground may be stated using 100 feet as a unit. The height of a person may be stated to the nearest half-inch, that is, using the half-inch as a unit. The result of measurement should be stated so as to indicate what unit was used but this is not always done. In this unit, a $2\frac{3}{4}$ in. implies that the unit used was $\frac{1}{4}$ inch, and that the result is stated to the nearest fourth-inch. The result, "$2\frac{3}{4}$ in." thus applies to any measurement which is within half of the unit on either side of the $2\frac{3}{4}$ inch mark; that is, which is more than $2\frac{5}{8}$ and less than $2\frac{7}{8}$ inches. We have used the term "greatest possible error" to refer to the amount by which the actual measurement may vary from the stated result. The use of the word "error" may cause some difficulty. It should be emphasized that the word is used here to mean that a measurement such as $2\frac{3}{4}$ inches represents any of a series of measurements ranging from $2\frac{3}{4} - \frac{1}{8}$ to $2\frac{3}{4} + \frac{1}{8}$. This meaning should be distinguished from the more familiar use of the term to mean mistakes in using the measuring instrument, mistakes in reading the scale, or mistakes resulting from use of a faulty instrument, such as a poorly marked ruler.

The two terms, "precision" and "accuracy" are sometimes confusing. In common usage, precision is the size of the unit of measurement used; the smaller the unit, the more precise the measurement. This means, of course, that the more precise the measurement, the smaller is the greatest possible error. Accuracy of a measurement is the percent the greatest possible error is of the measurement. That is, accuracy is the relative error.

In order to develop clearly the concepts of unit of measurement and greatest possible error, it is suggested that the students be given considerable practice in measuring lengths of lines to the nearest inch, nearest half-inch, nearest tenth of an inch, and so on. They should be asked to state the greatest possible error for measurements, and to indicate within what range a stated measurement must fall.

Students sometimes have difficulty in seeing why measurements with the same significant digits have the same accuracy, regardless of the unit of measure. A development along the following lines is sometimes helpful. Significant digits are defined as the digits in the numeral which show the number of units. The measurements below are analyzed to show the unit, the number of units, and the possible error.

3570 ft. Unit: 10 ft. No. of units: 357 Greatest possible error: 5 ft.

0.0357 ft. Unit: 0.0001 ft. No. of units: 357 Greatest possible error: 0.00005 ft.

Each of these measurements contains three significant figures. To determine the accuracy, we find the relative error, or percent of error. In the first case, we have the ratio $\frac{5}{3750}$. In the second case, the ratio is $\frac{0.00005}{0.0357}$, which equals $\frac{5}{3750}$.

Students also have difficulty in understanding why measurements with a larger number of significant digits have greater accuracy. An example similar to that above can be used.

3.57 ft. Unit: 0.01 ft. No. of units: 357 Greatest possible error: 0.005 ft.

35.70 ft. Unit: 0.01 ft. No. of units: 3570 Greatest possible error: 0.005 ft.

Relative error of 3.57 ft. is $\frac{0.005}{3.57}$, or $\frac{5}{35700}$.

Relative error of 35.70 ft. is $\frac{0.005}{35.70}$. or $\frac{5}{3570}$.

The student should constantly be cautioned that the rules presented for computation with approximate data are "rough" and are not universally applicable.

Bakst in his Approximate Computation* makes this important point:

> "Generally speaking, the technique of Approximate Computation is not mechanical. The performance of numerical processes may be thought of as mechanical, but the arithmetic of Approximate Computation can be fully appreciated if and only if the interpretative processes are predominant. Only when a student is conscious of the nature of the data and can interpret the approximativeness and the meaning of the numerical results obtained by him, will he understand the importance of Approximate Computation as a fundamental part of applied mathematics."

For instance, when we carry a computed number like π or a trigonometric function to one more significant digit than the least precise of the approximate factors, we are attempting to minimize the error introduced by the numbers arising from calculation rather than measurement.

If it seems desirable, this unit might be increased in scope by reports on such topics as the history of standard units of measure, various measuring devices used in industry and science, and the work of the United States Bureau of Standards.

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*Twelfth Yearbook of the National Council of Teachers of Mathematics, Washington, D. C.: National Council of Teachers of Mathematics, 1937. p. 18.

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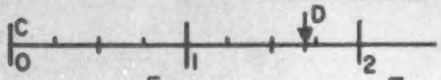
Answers to Exercises 10-1

1. 

$\overline{CD} = 1\frac{3}{4}$ inches.

2. Between $1\frac{5}{8}$ inches and $1\frac{7}{8}$ inches; $\frac{1}{8}$ inch from $1\frac{3}{4}$ inches.

3. The point D may be anywhere between $1\frac{5}{8}$ inches $(1\frac{3}{4} - \frac{1}{8})$ and $1\frac{7}{8}$ inches $(1\frac{3}{4} + \frac{1}{8})$.

4. (a) $\frac{1}{2}$ inch.

   (b) $1\frac{1}{4}$ inches and $1\frac{3}{4}$ inches.

   (c) $1\frac{1}{2} \pm \frac{1}{4}$ inches.

   (d) $\frac{1}{4}$ inch.

5. (a) Between $2\frac{9}{32}$ and $2\frac{11}{32}$.

   (b) $\frac{1}{32}$

6. $\frac{1}{16}$; $\frac{1}{16}$.

7. $\frac{1}{8}$ inch.

8. (a) $\frac{1}{10}$ cm.

   (b) $3\frac{7}{10} \pm \frac{1}{20}$ cm.

   (c) $\frac{1}{20}$ cm.

9. $\frac{1}{200}$ cm.

10. One-half of the unit used.

Answers to Exercises 10-2

1. 3.20 inches.

2. 4.0 inches.

3. (a) 5.2 feet.

   (b) 0.68 feet.

(c) Both have the same precision.

4. $12\frac{1}{2}$ and $13\frac{1}{2}$ years old.

5. (a) a. 100 feet
   b. 1 foot
   c. 10 feet
   d. 0.1 foot
   e. 0.0001 foot
   f. 0.1 foot

   (b) a. 50 feet
   b. 0.5 foot
   c. 5 feet
   d. 0.05 foot
   e. 0.00005 foot
   f. 0.05 foot

6. (a) e is the most precise.
   (b) a is the least precise.
   (c) Yes, d and f have the same precision.

7. (a) $420\underline{0}$
   (b) $23,\underline{0}00$
   (c) $48,0\underline{0}0,000$

8. (a) 2
   (b) 4
   (c) 1
   (d) 4
   (e) 4
   (f) 2
   (g) 4
   (h) 3

9. (a) 4
   (b) 4
   (c) 2
   (d) 3
   (e) 5
   (f) 2

Answers to Exercises 10-3

1. (a) 0.5 foot
   (b) 0.05 inch
   (c) 5 miles
   (d) 0.5 foot
   (e) 0.005 inches
   (f) 0.0005 foot
   (g) 500 miles
   (h) 5 miles

2. (a) 0.0096
   (b) 0.012
   (c) 0.0019
   (d) 0.0014
   (e) 0.00071
   (f) 0.083
   (g) 0.0093
   (h) 0.000093

3. (a) 0.05 foot; .54%
   (b) 0.0005 foot; .54%
   (c) 5 feet; .54%
   (d) 500 feet; .54%

4. The percents of error are the same for each measurement. In each case, the greatest possible error was the same fractional part of the measurement.

5. (a) 0.1 foot
   (b) 0.001 foot
   (c) 10 feet
   (d) 1,000 miles
   (e) 0.1 foot
   (f) 0.001 inch

6. (a) 3
   (b) 4
   (c) 3
   (d) 4

7. (a) 0.00096
   (b) 0.000096
   (c) 0.0014
   (d) 0.00014

8. Yes. As the number of significant digits increases the relative error decreases. The larger the number of significant digits the greater the accuracy.

9. 7,812 inches has the greatest accuracy.
   0.2 inch has the least accuracy.

10. (a) $36\frac{1}{2}$ inches, $22\frac{1}{4}$ inches, $46\frac{2}{7}$ inches, $32\frac{3}{8}$ inches, $27\frac{0}{16}$ inches.
    (b) 3 inches, 82.4 inches, 4.62 inches, 3.041 inches, 0.3762 inches.

*11. 6 feet ± $\frac{1}{2}$ foot.

$3\frac{1}{2}$ inches ± $\frac{1}{8}$ inch.

7.2 miles ± 0.05 mile.

3 yards 4 inches ± $\frac{1}{4}$ inch.

3.2 inches ± 0.005 inch.

12. (a) 4
(b) 4
(c) 5
(d) 2
(e) 4
(f) 2
(g) 5
(h) 4
(i) 5
(j) 2
(k) 2
(l) 8

13. (a) 4.63×10^8
(b) 3.27×10^5
(c) 4.62×10^{-4}
(d) 3.2004×10^1
(e) 2×10^0
(f) 4.00×10^{-5}
(g) 3.68×10^6
(h) 8×10^{-8}
(i) 7.2×10^{10}

14. e, j, i, c, g, a, f, b, d, h

15. BRAINBUSTER. $\frac{.0001}{3\frac{1}{2}} = \frac{10^{-4}}{3.5} = \frac{1}{3.5\times10^4}$

$$\frac{10^7}{8.6\times6\times10^{12}} = \frac{1}{51.6\times10^5} = \frac{1}{5.16\times10^6}$$

If two fractions have the same numerator, then the one with the larger denominator has the smaller value. Since, by definition, a measurement with a smaller relative error is the more accurate, the astronomer made the more accurate measurement.

Answers to Exercises 10-4

1. (a) $\frac{3}{4}$ inch.
(b) $\frac{7}{8}$ inch.
(c) 0.055 inch.
(d) 0.15 inch.
(e) 0.0510 inch.
(f) $\frac{7}{32}$ inch.

2. (a) 731.8

   (b) 145.1

   (c) 1,758.04

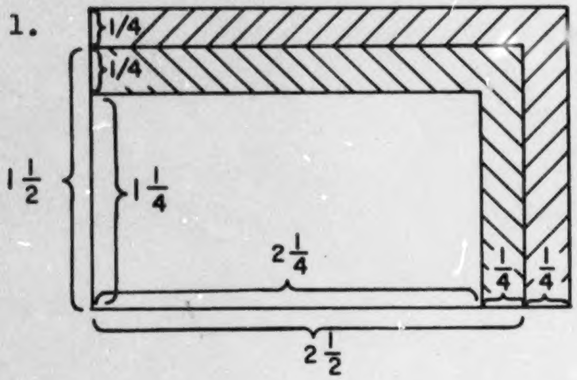
3. (a) 1.0

   (b) 734

   (c) 4780

Answers to Exercises 10-5

1.



Largest area = $4\frac{13}{16}$ sq. in.

Smallest area = $2\frac{13}{16}$ sq. in.

Difference = 2 sq. in.

Area is ($3\frac{3}{4}$ sq. in.) or 4 sq. in.

(To the nearest $\frac{1}{2}$ sq. in.)

2. (a) 150

   (b) 17,000

   (c) 11×10^4

3. (a) 4.4

   (b) 1.14

   (c) 3.92×10^5

4. 2.55×10^5 (or 255,000) square rods.

5. (a) 11 square inches.

   (b) 13.70 square feet.

   (c) 20 square miles.

6. 46.9 pounds. Three significant digits are used here because 75 is an exact counting number and does not affect the number of significant digits.

7. 250

8. (a) 13 feet (0.375 x 34)
   (b) 220 feet (3.27 x 67)

Sample Test Questions

PART I. TRUE - FALSE.

(T) 1. Counting separate objects is considered to be an exact measurement.

(T) 2. The smaller the unit, the more precise is the measurement.

(F) 3. The greatest possible error would be one-sixteenth inches if the length of a line is measured to the nearest $\frac{1}{4}$ inch.

(T) 4. The smaller the percent of error, the greater is the accuracy of the measurement.

(F) 5. If a measurement of a line is stated to be 10.0 inches, it implies that the line was measured to the nearest inch.

(F) 6. The more precise the measurement, the greater is the possible error.

(T) 7. A measurement of 300 miles has the same greatest possible error as a measurement of 700 miles.

(T) 8. There are four significant digits in the measurement 7,003 miles.

(T) 9. The term "greatest possible error" of a measurement does not refer to a mistake made in the measurement.

(F) 10. The greatest possible error of the sum of several approximate measurements is the same as the greatest possible error of the least precise measurement.

PART II. MULTIPLE CHOICE.

1. The most precise measurement is:

A. $26\frac{1}{2}$ inches.

*B. 26.0 inches.

C. 260 inches.

D. $26\frac{1}{4}$ inches.

2. The number with the greatest accuracy is the one with the least:

A. precision.

B. possible error.

*C. percent of error.

D. number of significant digits.

3. The greatest possible error in the sum of 45.5 in., 36.05 in., 5.1 in. is:

*A. 0.105 in.

B. 0.05 in.

C. 0.055 in.

D. 0.005 in.

PART III. COMPLETION.

1. The measurement of a line segment was stated to be $1\frac{1}{8}$ inches. The measurement might be stated as $1\frac{1}{8}$ ± $\frac{1}{(16)}$.

2. The greatest possible error in a measurement is always $\frac{1}{(2)}$ the unit used.

3. A measurement of $34\frac{1}{4}$ inches has the same precision as a measurement of $5\frac{0}{(4)}$.

PART IV. MISCELLANEOUS.

1. Measure the length of the line segment $\overline{AD}$ to the nearest eighth of an inch. A________________D ___($1\frac{4}{8}$)_______

2. Compute the percent of error in the measurement 25 inches. How would this compare with the percent of error in the measurement 25 miles?

   1. ($\frac{0.5}{25}$ = 0.02 = 2 percent)
   2. (Same)

UNIT 11

PERMUTATIONS AND COMBINATIONS

General Remarks

The study of permutations and combinations is a "fresh start" for many students because nearly all that is required as a background is a working knowledge of the four number operations and an understanding of literal numeration. It might be well to call this fact to the attention of certain pupils who may be able but not always willing.

Committee arrangements are exclusively used in most of the text material because the familiarity of pupils with this type of arrangement ensures that the only new ideas introduced in each section are those of a purely mathematical nature. Other varied and interesting applications of the mathematical principles learned in this unit are found in the exercises following "Class Discussion Exercises".

Pascal's Triangle, as developed in section 1, not only is an exciting application of combinations but also serves other purposes as well. It is an excellent device for helping student's recall quickly the number of combinations that can be formed from n things r at a time. Having this information about combinations readily available speeds up class discussion and allows for development of concepts without interrupting the train of thought with constant, somewhat irrelevant, calculations. Also the patterns involved in the triangle suggest certain inductive conclusions about the number relations among the various entries.

Several relations among the number facts in Pascal's Triangle are brought out. The sum of the numbers in any row is always a power of two; each entry is the sum of the two numbers nearest it on the preceding line; the greatest common factor in any prime-numbered row is the number of the row; the third term in each row can be predicted by using the formula, $\frac{n}{2}(n - 1)$; the symmetry of the triangle pictures the fact that, when a selection of r objects from n objects is made, another group of $(n - r)$ objects is selected automatically.

Contrary to what is probably the usual procedure, combinations are used as an introduction to permutations. By ordering the members of a subcommittee and applying the principles learned in the study of combinations, Property 1 is determined: If k is a counting number, then the number of permutations of any set of k numbers is $k!$. The study of permutations in this unit does not include those groups in which items are repeated such as the permutations for the word, Mississippi.

One more property is established: If in a set of pairs each possible left member appears in the same number of pairs as every other left member then the total number of pairs is the product of the number of possible left members and the number of pairs which have a given left member. Having made this conclusion about pairs, Property 2 is extended to include triples or larger groupings by the simple device of separating a triple into two members of which the left member is a pair. Property 2 is then applied to the pair of members comprising the triple.

In section 11-3, "combination" is formally defined and the corresponding symbol, $\binom{n}{r}$, is introduced. By applying both Properties 1 and 2, the equivalence of $\binom{n}{r}$ (r!) and $P_{n,r}$ is deduced along with $\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$

Each section contains a list of Class Discussion Exercises which are planned for developmental class discussions and are not assignment exercises. The intention is not to limit discussion to the specific questions listed. It is expected that the teacher will use this list of questions merely as a guide. However, each question is carefully placed for developmental purposes in the order given and is pointed toward a particular idea in the developmental sequence. If this material is relatively unfamiliar to the teacher it is recommended that the sequence be adhered to as given.

The exercises at the end of each part of a section are thought to be sufficient for one assignment, but if not, many similar problems can be found in high school second year algebra texts and also in many college freshman texts. Problems in these texts are not, in general, too difficult for students who have obtained a good grasp of the subject matter of the unit.

For better students, reference may be made to section 1 and to problem 6, exercise 3b of the supplementary unit on "Finite Differences".

It is hoped that the study of this unit will widen pupils' appreciation for mathematics as a science. The investigation of Pascal's Triangle begins with a nearly complete body of facts, includes the discovery and observation of quantitative relations, inductively suggests certain principles, verifies these deductively and derives essential generalizations. It is thought that the choice of Pascal's Triangle might be a good one because the body of facts is small, the quantitative relations are of a very simple nature, the discovery of the patterns involved is not difficult, and the number of experiments that may be performed is ample. The deductions associated with this investigation are of a very informal type and lend themselves readily to generalization.

It is also hoped that the study of this unit will prove to be a source of intellectual satisfaction and fun for the pupil and will help to strengthen his realization of the power of mathematics. He may be impressed by the vast amount of hard work that would be required to determine the third term in the two-thousandth line of Pascal's Triangle if he were actually to write down all the numbers in all the rows. He should be equally impressed by the ease with which the same information may be obtained by using $\frac{n(n - 1)}{2}$. The ability of the mathematician to predict the future with a maximum degree of certainty is another way in which the power of mathematics may be demonstrated. The formula above may be used by the student to make a prediction as to how many dots will have to be made to construct the one-thousandth diagram similar to those on page 13. This activity, while seemingly of little significance, is a parallel to the work of the mathematician who predicts that an airplane, still on the drawing boards, will, five years hence, carry a payload of five tons at a speed of 1650 miles per hour at an altitude of 10,000 feet and, at the end of the five year period, be shown correct in every detail. Mathematics, indeed, might be referred to as a "crystal ball"!

It is suggested that the following materials be kept close at hand during class discussions so that ready visual illustrations of various combinations and permutations may be given:

- A. Collections of objects that may be arranged according to color.
  - 1. Crayons.
  - 2. Flags (Boy Scout semaphore flags, yachting signal flags).
  - 3. Buttons, poker chips, or beads.
  - 4. Painted thumbtacks. (Can be set in a vertical board so that they can be seen without difficulty.)
  - 5. Geometric shapes cut from colored "construction paper". By using a paper cutter, many of these can be duplicated so that each member of the class may have some objects to arrange.
- B. Collections of objects that may be arranged according to shape.
  - 1. Geometric figures cut from "oak tag" paper.
  - 2. Models of the five regular polyhedrons.
  - 3. Letters of the alphabet made of paper or plywood.
  - 4. A flannel-board may be used to post the cut-outs where all can see. Since it is sometimes difficult to find colored material that will adhere to the board a suggestion would be to paste small pieces of sandpaper on the backs of the cut-outs.
- C. Collections of objects that may be arranged according to their measure.
  - 1. Coins, paper money.
  - 2. Weights of various denominations.

3. Balls of various diameters (marbles, golf balls, baseballs, etc.)
4. Dominoes.

Answers to Class Discussion Exercises 11-1a

1. {A, D, E}
2. Betty, Ellen, Alice
3. Yes
4. {B, A, E} ; {E, B, A}
5. {ABC}, (ABD}, {ABE}, {ACD}, {ACE}, {BCD}, {BCE}, {CDE}, {DEA}, {DEB},
6. 10
7. 6
8. Betty and Doris are on the same number of committees.
9. 4
10. $\frac{6}{10}$ or $\frac{3}{5}$
11. 3
12. $\frac{2}{6}$ or $\frac{1}{3}$
13. B, C, D
14. Alice and Doris
15. 10
16. 6
17. 4
18. {C,A} ; {C, B} ; {C, D} ; {C, e}

Answers to Class Discussion Exercises 11-1b

1. One committee with four members.
2. One committee with no members at all.
3. 1, 3, 3, 1
4. One committee with no members.
   Two committees with one member.
   One committee with two members.
5. 1, 1
6. 6
7. {E, A, B} and others. Two vacancies. Four.
8. Also 6.
9. 4
10. {A, B, C} and others. Three vacancies. 4
11. 3, 4

Answers to Class Discussion Exercises 11-1c

1. 1
2. 6
3. Yes, (1 + 5)
4. 1, 6, 15, 20, 15, 6, 1.

Answers to Class Discussion Exercises 11-1d

1. 10th row.
2. 15 + 14 + 13 + ... + 2 + 1.
3. 89 + 88 + 87 + ... + 2 + 1.
4. $$\begin{array}{l} 1 + 2 + 3 + \ldots 13 + 14 + 15. \\ \underline{15 + 14 + 13 + \ldots 3 + 2 + 1.} \\ 16 + 16 + 16 + \ldots 16 + 16 + 16 \end{array}$$
5. 15, 240.

6. 120
7. 120

Answers to Class Discussion Exercises 11-2a

1. 2
2.

Treasurer	Secretary
Betty	Ellen
Ellen	Betty

3. 2
4. 2
5. 6

Answers to Class Discussion Exercises 11-2b

1. $5! = (5)(4!)$
2. $4! = (4)(3!)$
3. $8! = (8)(7!)$
4. $436! = (436)(435!)$
5. $8429 = 1 + 8428$; $8429! = (1 + 8428)(8428!)$
6. k is one less than $(k + 1)$; $(k + 1)! = (k + 1)(k!)$

Answers to Class Discussion Exercises 11-2c

1. Vanilla ice cream with nut topping.
2. $\{C,M\}$; $\{C,N\}$; $\{S,M\}$; $\{S,N\}$; $\{V,M\}$; $\{V,N\}$.
3. 6
4. 3
5. 2
6. $6 = (3)(2)$
7. 10
8. 12

9. Package D is chosen for your friend and package A for you.

10. (B,D) means package B is for your friend and package D is for you.

    (D,B) means package D is for your friend and package B is for you.

11. Package C would be for your friend and also not for your friend.

12. {A,B} {B,A} {B,C} {C,B}
    {A,C} {C,A} {B,D} {D,B}
    {A,D} {D,A} {C,D} {D,C}

13. 12

14. 4

15. 4

16. 3

17. No

18. 12 = (4)(3)

19. Yes.

Answers to Class Discussion Exercises 11-2d

1. The order from top to bottom is different for each triple.
2. 5
3. 5
4. 4
5. (5)(4) = 20
6. 5
7. 3
8. No.
9. (20)(3) = 60

10. 60

Answers to Exercises 11-1a

1. (a) {A,B,C,D}, {A,B,C,E}, {A,B,D,E}, {A,C,D,E}, {B,C,D,E}.
   (b) 1
   (c) same
   (d) {A}, {B}, {C}, {D}, {E}.
   (e) 1
2. (a) 1
   (b) 4
   (c) {K}, {L}, {M}, {N}.
   (d) The three-member committee is composed of all the club members who are not in the corresponding one-member committee.
   (e) {L,M,N}, {K,M,N}, {K,L,N}, {K,L,M}.
   (f) {K,L}, {K,M}, {K,N}, {L,M}, {L,N}, {M,N}.
3. (a) If the club members are called P, Q, R, there is one committee with three members, namely {P,Q,R}; there are three committees with two members, namely {Q,R}, {P,R}, {P,Q}; there are three committees with one member, namely {P}, {Q}, {R}.
   (b) If U and V are the club members, the committees are {U,V}, {U}, {V}.
   (c) If the club member is Z, the only committee is {Z}.
4. 6
5. 10

Answers to Exercises 11-1b

2. The entry 6 may be interpreted as the number of possible committees with two members each in a club with four members. Suppose we let the club members be A,B,C,D. Every two-member committee either includes A or excludes A. A two-member committee including A is completed by the choice of one person from the remaining three; thus the number of two-member committees including A is the same as the number of one-member committees from the club {B,C,D}. This latter number is 3, the left-hand "3" in the third row of the table.

A two-member committee excluding A is determined by the choice of two persons from the remaining three; thus the number of two-member committees excluding A is the same as the number of two-member committees from the club {B,C,D}. This latter number is 3, the right-hand "3" in the third row of the table.

Since the two-member committees belong to one or the other classification, the total number of them is the sum of the numbers of the committees of each type: $6 = 3 + 3$.

3. The explanation is very similar to that given for Problem 2 and will not be repeated in detail. Here we are concerned with 3, the number of possible two-member committees in a three-member club. Suppose the committee members are A,B,C. Every two-member committee either includes or excludes A. Each type may be associated with a committee from the club {B,C}. The number of committees of the respective types is 2 and 1. The total is: $3 = 2 + 1$.

Answers to Exercises 11-1c

1. (a,b) The left hand column answers part (a), the right hand column part (b).

{A,B}	{C,D,E,F}
{A,C}	{B,D,E,F}
{A,D}	{B,C,E,F}
{A,E}	{B,C,D,F}
{A,F}	{B,C,D,E}
{B,C}	{A,D,E,F}
{B,D}	{A,C,E,F}
{B,E}	{A,C,D,F}
{B,F}	{A,C,D,E}
{C,D}	{A,B,E,F}
{C,E}	{A,B,D,F}
{C,F}	{A,B,D,E}
{D,E}	{A,B,C,F}
{D,F}	{A,B,C,E}
{E,F}	{A,B,C,D}

(c) Yes; 15 = 10 + 5; (the "10" is the right-hand entry and the "5" is the right-hand one in the fifth row).

(d) There are 20 of them; the following is an incomplete list: {A,B,C}, {A,B,D}, {A,B,E}, {A,B,F}, {A,C,D}, etc.

(e) Yes; 20 = 10 + 10.

2. 1 7 21 35 35 21 7 1

3. 1 8 28 56 70 56 28 8 1

4. 1 23

5. 57 1

6. (a) 8; 16; 32.

(b) 2^3; 2^4; 2^5.

(c) The sums are successive powers of 2; each is twice its predecessor.

(d) Predict 2^6.

(f) 2^7 predicted; agrees with 128.

(g) 2^{12}

(h) 201; 2^{200}

Answers to Exercises 11-1d

1. 435

2. (a) $n - 1$

(b) $\underline{2 + 3 + \cdots + (n - 3) + (n - 2) + (n - 1)}$

(c) $\underline{n + n + \cdots + n + n + n}$

(d) $n - 1$ (same as answer to part (a).)

(e) $(n - 1)\cdot n$

(f) $\frac{n(n - 1)}{2}$

3. 105

*4. (a) 7

(b) 5

(c) 3

(d,e,f) Several possible beliefs may be suggested; most of them will need to be discarded by testing in other rows. The principle desired is: If p is a prime number, then all the numbers in the p^{th} row of the Pascal triangle (excluding the ones at the end) are divisible by p, and furthermore their greatest common factor is p itself.

*5. (a) 1, 3, 6, 10, 15, 21.

(b) These are the numbers studied in the test: the third entries in successive rows (after the first) of the Pascal triangle.

(c) $\frac{81 \cdot 80}{2}$ or 3240 (note no third entry in first row of the Pascal triangle.)

(d) $\frac{1001 \cdot 1000}{2}$ or 500500.

(e) Bowling.

Answers to Exercises 11-2a

1. (a) Chairman A, Secretary F, Treasurer H;
Chairman A, Secretary H, Treasurer F;
Chairman F, Secretary H, Treasurer A;
Chairman F, Secretary A, Treasurer H;
Chairman H, Secretary A, Treasurer F;
Chairman H, Secretary F, Treasurer A.

(b) 6

(c) Yes: three alternatives are available for Chairman, and with each of these possibilities there are two alternatives for distributing the other offices.

2. (a) Three jobs and three people.

(b) 6

(c) 6

(d) 6

(e) 4

(f) 24

(g) Same.

(h) If we begin the analysis by focussing attention on one particular office, there are 4 alternatives for the person to fill the office.

(i) Yes: compare the answers to Problems 1(c) and 2(h).

3. (a) First Ron, second Sam, third Ted; first Ron, second Ted, third Ron; first Sam, second Ted, third Ron; first Sam, second Ron, third Ted; first Ted, second Ron, third Sam; first Ted, second Sam, third Ron.

(b) 6

(c,d) (Completed tables not reproduced here.)

(e) Explanation very similar to that in answer to Problem 1(c).

4. 6

5. 24

6. 23

7. 120

8. 24

Answers to Exercises 11-2b

1. 3!

2. 720

3. 14 (note that 14! is the product of 14 and 13!)

4. $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6 \cdot 5 \cdot (4 \cdot 3 \cdot 2 \cdot 1) = 6 \cdot 5 \cdot (4!)$

5. 8!

6. 60! is the product of all counting numbers from 1 to 60 inclusive; when this number is multiplied by 61 and by 62, the new product has, as factors, all the counting numbers from 1 to 62 inclusive; thus the new product is 62!.

7. 9!

8. 8!

9. 7!

10. 8!

Answers to Exercises 11-2c

1. 28

2. 15

3. 676

4. 2450

5. 30

6. (a) 72

   (b) 56

   (c) 36 (note that this is half the number in part (a).)

   (d) 756; 702; 378.

Answers to Exercises 11-2d

1. 12

2. $P_{27,19}$

3. $20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15$

4. $P_{49,7}$

5. (a) $12 \cdot 11 \cdot 10$

   (b) $(12 \cdot 11 \cdot 10)(9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$

   (c) 12!

   (d) Subtraction

   (e) $\frac{12!}{9!}$

6. (a) $20 \cdot 19 \cdot 18 \cdot 17$

(b) $(20\cdot 19\cdot 18\cdot 17)(16\cdot 15\cdot 14\cdot \cdots \cdot 3\cdot 2\cdot 1)$

(c) $20!$

(d) Subtraction

(e) $\frac{20!}{16!}$

*7. (a) $n(n-1)\cdots(n-r+1)$

(b) $(n-r)!$

(c) $n!$

(d) $\frac{n!}{(n-r)!}$

8. (a) 117600

(b) 250,000

9. (a) 10^5

(b) $P_{10,5}$

10. 20

*11. 8

Answers to Exercises 11-3a

1. (a) $\binom{12}{7}$

(b) $P_{12,7}$

(c) $\binom{m}{3}$

(d) $\binom{n+2}{k}$

2. The number of combinations of 6 objects taken 2 at a time; the number of permutations of 8 objects taken 4 at a time; fifty-two factorial; the number of combinations of 52 things 13 at a time; the number of permutations of 9 things 7 at a time; eight fourths.

3. (a) 15; 15
   (b) 35; 35
   (c) 56; 56
   (d) 3; 3
   (e) 28; 28

*4. (b) $\binom{s}{a} = \binom{s}{b}$

   (c) The selection of a committee with a members from a club with s members essentially separates the club into two parts, one with a members and the other with b members. Thus the number of committees with a members is the same as the number of committees with b members.

5. If n is any counting number, then $\binom{n}{n} = 1$.

6. If n is any counting number, then $\binom{n}{0} = 1$.

7. This is Problem 2(f) in Exercises 11-1d, stated in the new notation.

Answers to Exercises 11-3b

1. 120
2. $\binom{15}{5}$
3. 56
4. (a) 66
   (b) 132
5. 63 (note that $2^6 - 1 = 63$.)
6. 60
7. 140
8. 30240 (in other words, more than three years)
9. 36

10. 10 weeks and 2 days more.

11. $\binom{52}{13}$

12. 40320

*13. Since $P_{n,r} = \frac{n!}{(n-r)!}$ and since $\binom{n}{r}$ is the quotient of $P_{n,r}$ by $r!$, we obtain $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$.

UNIT 12

PROBABILITY

12-1. Chance Events

(Recommended teaching time $1\frac{1}{2}$ - 2 days)

In section 1 we attempt to define "Chance events" and use this definition to help the students learn what is meant by probability. The statements in the opening paragraph should serve to introduce to the students the difference between what is stated and what is likely to occur. We assume that events likely to occur are not influenced by external factors.

The uses of probability are touched upon lightly. This area can be expanded in class discussion. Such discussions may serve as motivating devices if not extended too long. The students will probably have more interest in actually working problems than in lengthy discussion of applications of probability.

The pupils should be encouraged to construct all tables, or models, used in the discussion and the problems in the exercises. Suggestions for easy ways to make the tables are made in the next section. However, if students are encouraged to make their own modesl, such as cubes, wheels with pointers for spinning, and so on, for experimenting, they should understand that such models are not necessarily "honest". Whenever a model is referred to in an example or problem, it is always assumed to be honest.

The measure of chance is first introduced as the number of true outcomes as compared to the total number of possible outcomes. Then the use of a number to represent the measure is introduced. It may be pointed out in class discussion that when the probability is $\frac{2}{4}$, or some similar fraction, the usual procedure is to reduce the fraction to lowest terms.

The notation P(A) is introduced. Whenever possible this notation should be used so that the students gain familiarity with it. It is important that the teacher exercise correct terminology whenever possible to instill in the students the importance of doing so. Teachers should not, however, insist on exacting notations or terminology if by doing so the student is distracted from an important idea.

The notion that the "law of averages" affects outcomes may be difficult to dispel. Burton Jones' book, *Elementary Concepts of Mathematics*, pages 209-10 is an excellent teacher reference on this matter. Also, Horace C. Levinson's, *The Science of Chance*, discusses this topic.

It is important that the students understand that each outcome listed in the tables in this section is considered equally likely to occur. The tables aid us in counting the possible outcomes. The student should also gain some understanding of classifying procedures used.

Answers to Exercise 12-1

1.

Possible outcomes	Occurence of the Event	
B_1 B_2	O	
B_1 W	X	
B_2 W	X	$P = \frac{2}{3}$
B_2 W		

2. $P = \frac{1}{3}$

3. Possible Outcomes

	1st coin	2nd coin	3rd coin	Occurence of the Event
*	H	H	H	X
	H	H	T	O
	H	T	H	O
	H	T	T	O
	T	H	H	O
	T	H	T	O
	T	T	H	O
	T	T	T	O

* H means heads and T means tails.

$$P = \frac{1}{8}$$

4. Since there are 3 possible outcomes for one tail and two heads, $P = \frac{3}{8}$. Students may list the outcomes of the statement in order;

O, X, X, O, X, O, O, O .

5. Label the hats A, B, C. Assume you want hats marked A and B.

Possible Outcomes	Occurrence of Event	
A B	X	
A C	O	$P = \frac{1}{3}$
B C	O	

Note: The following questions may be discussed in class,

(a) What is the chance that you will pick your own hat if you pick up two hats? $(P = \frac{2}{3})$

(b) What is the chance that you will pick one of the two wanted hats if you pick two hats? (P = 1 or it may be pointed out that the outcome is not left to chance.)

6. There are 13 spades. $P = \frac{13}{52}$ or $\frac{1}{4}$

7. There are 4 aces. $P = \frac{4}{52}$ or $\frac{1}{52}$

8. There is only one ace of spades. $P = \frac{1}{52}$

9. (a) $P = \frac{4}{53}$

(b) The total number of cards in the deck is not the same in each problem.

10. (a) 48

(b) 40

(c) $P = \frac{40}{48} = \frac{5}{6}$

11. $P = \frac{8}{48} = \frac{1}{6}$

12. (a) No

(b) 0 any possible outcome must be assigned a measure greater than 0.

13. (a) $P = \frac{1}{8}$

(b) $P = \frac{1}{8}$

(c) $P = \frac{3}{8}$

(d) $P = \frac{3}{8}$

(e) No

(f) No

(g) The sum is 1. Since we are adding the measures of all possible outcomes the sum must be one. We assume that all outcomes are equally likely to occur and assign measures of chance on this basis.

14. (a) $P = \frac{1}{5}$

(b) $P = \frac{1}{4}$

(c) No. The total number of cards is not the same.

(d) $P = \frac{1}{3}$

(e) The value of P increases as the total number of cards decreases.

15. (a) $P = \frac{1}{2}$

(b) The outcome of an event in which $P < \frac{1}{2}$ is not likely to occur. We can say that if a large number of such cases are considered we can expect less that than $\frac{1}{2}$ to occur.

(c) The outcome of an event in which $P > \frac{1}{2}$ is likely to occur. The chances are "better than even" that the outcome will occur.

16. (a) No. We should expect only 4-6 heads to show according to the measure of chance, in this case $P = \frac{1}{2}$.

(b) The measure of chance is $\frac{1}{2}$ for each separate toss.

(c) None. P for each toss remains the same.

12-2. Probability from Models

(Recommended teaching time $1\frac{1}{2}$ - 2 days.)

Section 2 introduces the formula $T = S^n$. By relating the development of the formula to the tables the students made in Exercise 12-1, most students should not experience difficulty in using the formula. The first 4 problems in Exercise 12-2 may be solved by using this formula.

The pattern in the truth tables should be evident after comparison and discussion. Levinson discusses this briefly in his book, The Science of Chance, pages 59-60. It was felt that the use of dice would be left to the teacher's discretion. (A cube with sides, numbered 1, 2, 3, 4, 5, and 6 or lettered A, B, C, D, E, and F serves the purpose equally well. Different colors applied to each face may also be used.) It may be that "tossing coins" may meet with some objections. Disks with different colored sides may be used instead of coins. It should be stressed that games of chance are used only because they serve as good models for discussing elementary probability. Interest in gambling led to early interest in probability.

The tables for the weather forecasts serve to show analyses of a different kind of data. Here all possible outcomes are not listed listed. A more subtle property is not discussed, namely that while we assume that each outcome is equally likely this is not always possible in events such as the one discussed here. It was considered prudent to avoid any reference to this discussion.

An effort is made to relate the use of probability to the occurrence of mass events. The term "mass event" is used for lack of better term. Perhaps teachers should not stress the use of the term "mass" if it leads to complications in later situations. The student may gain some understanding needed for accuracy in estimating the measure of probability. The measure of probability cannot influence the outcome. It is only a tool used in predicting possible outcomes. "Luck" may certainly influence the tossing of a coin a few times, but in the long run measure of chance will hold up.

Answers to Exercise 12-2

1. $T = s^n$

   $T = 2^5$

   $T = 32$

2. (a) 4

   (b) $T = 4^2$ or 16

   (c) $T = 4^3$ or 64

   (d) $P = \frac{1}{4}$

3. (a) $T = 6^1$ or 6

   (b) $T = 6^2$ or 36

   (c) $P = \frac{1}{6}$

   (d) $P = \frac{1}{36}$

4. (a) $T = 8^1$ or 8

   (b) $T = 8^2$ or 64

5.

Days	Occurrence of Event	
	A	B
1	X	0
2	X	0
3	0	X
4	0	X
5	0	X
6	X	0
7	X	0
8	0	X
9	0	X
10	0	X
11	0	0
12	0	0
13	X	0
14	X	0
15	0	X
16	0	X
17	0	X
18	0	X
19	X	0
20	0	X

$P(A) = \frac{7}{20}$

$P(B) = \frac{11}{20}$

Sunshine seems more likely. Note that on cloudy days with no rain both statements are false.

6. $P = \frac{5}{35} = \frac{1}{7}$

7. (a) $P = \frac{1}{2}$

   (b) about 25

8. (a) $P = \frac{1}{2}$

   (b) $P = \frac{1}{3}$

   (c) $P = \frac{1}{4}$

9. (a) $\frac{1}{52}$

   (b) $\frac{4}{52}$ or $\frac{1}{13}$

10. $\frac{5}{12}$

11. $T = 2^6$ or 64

12. $P = \frac{2}{3}$

13. $\frac{2}{3}$ (if you include the possibility of standing between them. $P = \frac{1}{2}$ if you do not include this possibility.)

14. $P = \frac{500}{725}$ or $\frac{20}{29}$

15. $P = \frac{1}{64}$

12-3. Mutually Exclusive Events

(Recommended teaching time 1 or 2 days)

Here the idea of "mutually exclusive events" is very important. It is enlightening to compute the probability in two different ways in a number of examples: by dividing the number of favorable events by the number of events, and by adding the probabilities.

Answers to Exercise 12-3.

1. (a) Mutually exclusive.
   (b) Not mutually exclusive since the two events can occur at the same time.
   (c) Not mutually exclusive.
   (d) Mutually exclusive.
   (e) Not mutually exclusive.
   (f) Mutually exclusive.
   (g) Mutually exclusive.
   (h) Mutually exclusive.
   (i) Not mutually exclusive. Your mother may also be your teacher.
   (j) Mutually exclusive.

2. P(A or B or C or D) = P(A) + P(B) + P(C) + P(D).
   Event A: Stops at 1; Event C: Stops at 5;
   Event B: Stops at 3; Event D: Stops at 7.
   $P(A) = \frac{1}{8}$; $P(B) = \frac{1}{8}$; $P(C) = \frac{1}{8}$; $P(D) = \frac{1}{8}$
   Therefore, P(A or B or C or D) $= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

*3. P(A or B) = P(A) + P(B)
   Event A: Stop at 2
   Event B: Stop at 4
   $P(A) = \frac{1}{2}$ (Since 2 is $\frac{1}{2}$ of the face)
   $P(B) = \frac{1}{6}$
   P(A or B) $= \frac{1}{2} + \frac{1}{6} = \frac{5}{6}$

4. (a) P(A or B) = P(A) + P(B)
   Event A: a 1 appears up.
   Event B: a 6 appears up.

$P(A) = \frac{1}{6}$; $P(B) = \frac{1}{6}$

$P(A \text{ or } B) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

(b) $P\ (2 \text{ or } 3 \text{ or } 4 \text{ or } 5) = \frac{4}{6} \text{ or } \frac{2}{3}$

5. (a) Find the probability of picking a white ball.
There will be 8 ways of picking exactly one white ball. Each way is assigned the probability $\frac{1}{10}$. Therefore,

$$P\ (\text{picking a white ball}) = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{8}{10} = \frac{4}{5}$$

The problem could also be solved by finding the probability of picking a red ball in one draw and then subtract this from 1. The teacher should emphasize that there might be many ways of solving the problem.

(b) $P\ (\text{picking a red ball}) = \frac{1}{5}$

6. (a) $\frac{1}{3} + \frac{2}{3} = 1$

The probability that an event happens or it doesn't happen is 1. An event happens or it doesn't happen.

(b) 1.

(c) Same as in (a) above.

$1 - \frac{1}{3} = \frac{2}{3}$: Answer for 4 b.

$1 - \frac{4}{5} = \frac{1}{5}$: Answer for 5 b.

7. This question asks for a general formula for the examples discussed in Exercise 6.
$P(A \text{ or } B) = P(A) + P(B) = 1$, where event B is the event "A does not occur."

8. (a) $P(A \text{ or } B) = P(A) + P(B)$

Event A : One red ball is drawn.

Event B : The other red ball is drawn.

$P(A) = \frac{1}{5}$; $P(B) = \frac{1}{5}$

$P(A \text{ or } B) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$

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(b) A: A red ball is drawn.

B: The other red ball is drawn.

There are now 4 possible events since we know that one white ball is drawn out and not replaced.

$P(A) = \frac{1}{4}$; $P(B) = \frac{1}{4}$

Therefore: $P(A \text{ or } B) = \frac{1}{2}$.

12-4. Independent Events.

(Recommended teaching time 1 or 2 days)

Here the contrast between mutually exclusive and independent events must be emphasized. The former means that if A occurs, then B cannot; while the latter means that if A occurs, B may or may not occur - furthermore, the occurrence of A has no effect on the occurrence of B.

Answers for Exercise 12-4.

1. (a) Yes. The event of a head coming up on the first toss has no influence upon the event of a head coming up on the second toss.

   (b) $P(A \text{ and } B) = P(A) \cdot P(B)$

   Event A : A head on the first toss

   Event B : A head on the second toss

   $P(A) = \frac{1}{2}$; $P(B) = \frac{1}{2}$

   $P(A \text{ and } B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

2. (a) $P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B) \cdot P(C)$ where A, B, and C are independent events.

   This could be generalized to any number of independent events. The general property is that multiplication can be used to compute the probability of independent events.

   (b) $P(9 \text{ heads or } 9 \text{ successive tosses}) = (\frac{1}{2})^9$ or $\frac{1}{512}$

3. (a) Not independent since the first draw not being replaced changes the probability of the follow draws.

   (b) Independent.

   (c) Not independent. Going to school has no influence upon becoming a lawyer.

   (d) Independent.

   (e) Not independent.

4. (a) $P(A \text{ and } B) = P(A) \cdot P(B)$

   Event A : Win over A ; Event B : Win over B .

   $P(A) = \frac{1}{2}$; $P(B) = \frac{2}{5}$

   $P(A \text{ and } B) = \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5}$

(b) It depends whether your team played A first or B first. If A was played first the probability of winning the second game is $\frac{2}{5}$ and if B was played first the probability of winning the second game would be $\frac{1}{2}$.

5. (a) Event A : Pointer of dial A stops on red.
Event B : Pointer of dial B stops on red.
$P(A) = \frac{1}{2}$; $P(B) = \frac{1}{2}$
$P(A \text{ and } B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

(b) Event A : Pointer of dial A stops on green.
Event B : Pointer of dial B stops on green.
$P(A) = \frac{1}{4}$; $P(B) = \frac{1}{6}$
$P(A \text{ and } B) = \frac{1}{4} \cdot \frac{1}{6} = \frac{1}{24}$

(c) Event A : Pointer of dial A stops on white.
Event B : Pointer of dial B stops on blue.
$P(A) = \frac{1}{4}$; $P(B) = \frac{1}{6}$
$P(A \text{ and } B) = \frac{1}{24}$

6. Event A : Throw a four.
Event B : Draw a spade.
$P(A) = \frac{1}{6}$; $P(B) = \frac{1}{4}$
$P(A \text{ and } B) = \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24}$

7. Event A: Draw a white ball.
Event B: Draw another white ball.
$P(A) = \frac{4}{9}$; $P(B) = \frac{4}{9}$
$P(A \text{ and } B) = \frac{4}{9} \cdot \frac{4}{9} = \frac{16}{81}$

8. Event A : Draw a white ball.
Event B : Drawn another white ball without replacing the first.
$P(A) = \frac{4}{9}$
$P(B) = \frac{3}{8}$ (The white ball was not replaced)

$P(A \text{ and } B) = \frac{4}{9} \cdot \frac{3}{8} = \frac{1}{6}$

This problem could also have been solved by using combinations as follows:

The number of ways A can occur is $\binom{4}{2} = 6$.

the number of ways 2 balls can be drawn is $\binom{9}{2} = 36$.

Thus, P(both balls being white) $= \frac{6}{36} = \frac{1}{6}$

9. Use combinations.

How many committees of this size could be chosen? This is $\binom{20}{3} = 1140$.

There is exactly one way to select a committee which has three members, you and two of your friends.

Thus, $P(A) = \frac{1}{1140}$

It is very unlikely that you will serve on such a committee chosen at random.

10. Event A : The first man will not solve the problem.

Event B : The second man will not solve the problem.

$P(A) = 1 - \frac{2}{3} = \frac{1}{3}$; $P(B) = 1 - \frac{5}{12} = \frac{7}{12}$

$P(A \text{ and } B) = \frac{1}{3} \cdot \frac{7}{12} = \frac{7}{36}$

The chance that the problem will be solved is $1 - \frac{7}{36} = \frac{29}{36}$

11. When 6 coins are tossed there are 2^6 ways the event (heads or tails) may occur. The easiest way to solve the problem is to first find the probability of not obtaining at least one or more heads. Since there is only one way that this may happen (that is, all six are tails), P(not getting at least 1 head)= $\frac{1}{26} = \frac{1}{64}$. Then the desired probability is: $1 - \frac{1}{64} = \frac{63}{64}$

Thus, P(getting at least 1 head) $= \frac{63}{64}$

12. BRAINBUSTER

Method I.

(a) This is a problem where replacements have not been made. Use combinations.

There are $\binom{5}{2}$ ways of drawing even numbers.

There are $\binom{10}{2}$ ways of drawing any kind of pair.

Thus, $P(2 \text{ even numbers}) = \frac{\binom{5}{2}}{\binom{10}{2}} = \frac{10}{45} = \frac{2}{9}$

Method II.

Using $P(A \text{ and } B) = P(A) \cdot P(B)$

$P(A) = \frac{1}{2}$; $P(B) = \frac{4}{9}$

$P(A \text{ and } B) = \frac{1}{2} \cdot \frac{4}{9} = \frac{2}{9}$

(b) The sum of two even numbers is an even number therefore the information in Part (a) is a start. The sum of two odd numbers is also an even number. Therefore we need pairs of odd numbers. There are $\binom{5}{2}$ = 10 pairs of odd numbers. The probability of drawing two odd numbers is the same as drawing two evens, that is, $\frac{2}{9}$.

Since we can satisfy the requirements of the problem either by drawing two even or two odds we add the probabilities

Thus, P(that the sum of the two numbers is even) =

$$\frac{2}{9} + \frac{2}{9} = \frac{4}{9}$$

This naturally could have been solved by finding the number of ways of obtaining a sum which was odd.

The P(of obtaining a sum whish is odd) = $\frac{25}{45}$

(Each odd number - there are 5 - must be matched with an even number - there are 5. Thus, the numerator is 5 x 5 = 25).

(c) List all the ways that the sum of the two numbers is divisible by 3.

1 and 2
1 and 5
1 and 8
2 and 4
3 and 6
2 and 7
4 and 5

4 and 8
3 and 9
2 and 10
5 and 7
5 and 10
6 and 9
7 and 8
8 and 10

P(That the sum is divisible by 3) = $\frac{15}{45} = \frac{1}{3}$

(d) P (The sum is less than 20) = 1

(e) P(The pairs have a sum more than 20) = 0

Note:

(d) Cannot fail and (e) cannot happen.

13. (a) Numerator: There is exactly 1 way to get four aces. The fifth card can be one of 48. Thus, there are 48 hands which could have 4 aces.

Denominator: How many possible 5 card hands could be drawn? This is $\binom{52}{5}$.

P(4 aces out of 5 cards) = $\frac{48}{\genfrac{}{}{0pt}{}{52}{5}}$

If the pupil wishes to calculate this, here is a start:

$$\frac{1}{(13)(17)(5)(49)}$$

This event is not very likely to occur by chance.

(b) There are 4 aces, 4 kings, 4 queens, 4 jacks, and 4 tens. Therefore, there are 4^5 ways of obtaining 1 ace, 1 king, 1 queen, 1 jack, and 1 ten.

P(A) = $\frac{4^5}{\genfrac{}{}{0pt}{}{52}{5}}$

It is not necessary to have the pupil do the computation unless he is interested in the fractional or decimal value.

14. (a) The two heads could be: $\genfrac{}{}{0pt}{}{4}{2} = 6$

Nickel - dime
Nickel - penny
Nickel - quarter
Penny - dime
Penny - quarter
Quarter - penny.

Since there are 3 pairs which contain a dime,

$P(A) = \frac{3}{6} = \frac{1}{2}$

(b) The three heads could be: $\binom{4}{3} = 4$

Dime - Quarter - Nickel
Dime - Quarter - Penny
Quarter - Nickel - Penny
Dime - Nickel - Penny

Since 3 contain a dime, $P(A) = \frac{3}{4}$

15. (a) List events:

Dime - Quarter - Nickel
Dime - Quarter - Penny
Dime - Quarter - Half

There are $\binom{5}{3} = 10$ (ways of getting 3 heads)

$P(A) = \frac{3}{10}$

(b) List events:

Dime - nickel
Dime - quarter
Dime - half
Dime - penny

There are $\binom{5}{2} = 10$. There are 10 ways of getting 2 heads.

$P(A) = \frac{4}{10} = \frac{2}{5}$

(c) Use, $P(A \text{ and } B) = P(A) \cdot P(B)$

Event A : two heads appear out of 5 coins
Event B : One of the two is a dime.

$$P(A) = \frac{\genfrac{}{}{0pt}{}{5}{2}}{2^5} = \frac{10}{32} = \frac{5}{16}$$

$P(B) = \frac{4}{\binom{5}{2}} = \frac{2}{5}$

$P(A \text{ and } B) = \frac{5}{16} \cdot \frac{2}{5} = \frac{1}{8}$

(d) Use, $P(A \text{ and } B) = P(A) \cdot P(B)$

Event A : 3 coins come up heads

Event B : 2 of these are a dime and a quarter.

$P(A) = \frac{\binom{5}{3}}{2^5} = \frac{10}{32} = \frac{5}{16}$

$P(B) = \frac{3}{10}$ (See 15a)

$P(A \text{ and } B) = \frac{5}{16} \cdot \frac{3}{10} = \frac{3}{32}$

16. List the possible lengths for a triangle:

10, 9, 8
10, 9, 7
10, 9, 6
10, 9, 5
10, 9, 4
10, 9, 3
10, 9, 2

10, 8, 7
10, 8, 6
10, 8, 5
10, 8, 4
10, 8, 3

10, 7, 6
10, 7, 5
10, 7, 4

10, 6, 5

9, 8, 7
9, 8, 6
9, 8, 5
9, 8, 4
9, 8, 3
9, 8, 2

9, 7, 6
9, 7, 5
9, 7, 4
9, 7, 3

9, 6, 5
9, 6, 4

8, 7, 6
8, 7, 5
8, 7, 4
8, 7, 3
8, 7, 2

8, 6, 5
8, 6, 4
8, 6, 3

8, 5, 4

7, 6, 5
7, 6, 4
7, 6, 3

7, 5, 4
7, 5, 3

6, 5, 4
6, 5, 3
6, 5, 2

6, 4, 3

5, 4, 3
5, 4, 2

3, 4, 2

There are 50 such combinations.

$$P(A) = \frac{50}{\genfrac{}{}{0pt}{}{10}{3}} = \frac{50}{120} = \frac{5}{12}$$

GLOSSARY OF TERMS

Estimate: A guess of approximation that an event will occur.

Forecast: A statement which is the best available estimate of what is expected to happen

Statement: A statement is a sentence which we can say is either true or false.

Truth Value: The truth value of a statement indicates whether the statement is true or false.

Chance: The probability or likelihood of an event happening which occurs due to outside forces.

Measure of Chance: A numerical expression that indicates the probability of an event occurring.

Formula for the Measure of Chance (p): Measure of Chance = p = $\frac{t}{s}$ =

$$\frac{\text{The number of times the statement is true}}{\text{The total number of times the statement is true or false}}$$

Best Estimate: The best forecast, approximation, or guess which you can make of an event occurring based on the data available.

Likelihood: The chance or probability of an event occurring.

Model for Chance Statement: A method of determining the measure of chance by listing all possible events, as in a table.

Suggested Test Items

I. True-False

(T) 1. If the probability of winning the football game next Friday is $\frac{3}{5}$ the team is more likely to win than to lose.

(F) 2. Probability is the ratio of the true statements compared to the false statements.

(T) 3. If a box contains two red and two black marbles, the probability of picking a red marble is the same as the probability of picking a black marble on the first draw.

(F) 4. If a penny is tossed 20 times, we expect to have heads show exactly 20 times.

(T) 5. It is known that neither of two students was born on February 29. The chance that the two have the same birthday is $\frac{1}{365}$.

(T) 6. If 100 pennies are tossed, we can expect exactly two heads to show about 25 times.

(F) 7. The truth value of a statement may be measured by the numbers $\frac{1}{2}$.

(T) 8. The probability of "X" winning the race is $\frac{3}{7}$. "X" has less than an even chance of winning.

(F) 9. "Truth Value" and "measure of chance" mean the same thing.

(T) 10. Decision making can be based on a knowledge of chance.

(T) 11. A statement such as a weather forecast is a chance statement.

(F) 12. An honest coin is one that when tossed and allowed to fall freely will always show heads.

(T) 13. Probability is sometimes used by governments.

(F) 14. To check the truth value of yesterday's weather forecast the measure of chance must be known.

(F) 15. If 100 pennies are tossed, we can expect that two heads will show exactly 25 times.

(F) 16. If two coins are tossed, three outcomes are possible: two heads, two tails, a head and a tail. Thus the probability of tossing two heads is one-third.

(T) 17. If five coins are tossed, the probability that two are heads is the same as the probability that three are heads.

(F) 18. Drawing a spade and drawing an ace from a deck of cards are mutually exclusive events.

(F) 19. If events A and B are mutually exclusive, the probability of the event: A and B , is the product of the probability of A and the probability of B .

(T) 20. If A and B cannot both happen they are not independent events.

(F) 21. If A and B can both happen they are independent events.

II. Multiple Choice.

(D) 1. Suppose you toss a coin five times and each time it comes up heads. The probability that a head will show on the next toss is . . .

A. $\frac{5}{6}$

B. $\frac{1}{6}$

C. $\frac{1}{5}$

D. $\frac{1}{2}$

E. None of the above is correct.

(A) 2. Mark has three cards. One card has the letter A printed on both sides, another the letter C on both sides, and the third has T on both sides. If he tosses them all on the floor at the same time, what are the chances that they will spell cat? (Assume that a card does not cover one of the others).

A. $\frac{1}{6}$

B. $\frac{1}{3}$

C. $\frac{1}{2}$

D. $\frac{3}{8}$

E. None of the above is correct.

(C) 3. A weatherman's forecasts are correct 20 times in one month and not correct 10 times in the same month. Based on this information, the estimate of the measure of chance that his future forecasts will be correct is . . .

A. $\frac{1}{3}$

B. $\frac{1}{2}$

C. $\frac{2}{3}$

D. $\frac{3}{4}$

E. None of the above is correct.

(B) 4. A drawer contains 10 blue stockings and 10 red stockings. What is the smallest number of stockings you need to take to be sure you have a pair of the same color stockings?

A. 2

B. 3

C. 5

D. 10

E. 11

(A) 5. There are two roads from Milltown to Luck and three roads from Luck to Turtle Lake. How many different ways can John travel from Milltown to Turtle Lake?

A. 6

B. 5

C. 3

D. 2

E. None of these.

(C) 6. The measure of probability may be found by using the formula

A. $P = \frac{1}{2}(E)$

B. $P = \frac{t}{s}$

C. $P = S^n$

D. $P = AB$

E. None of the above is correct.

(E) 7. If six honest coins are tossed and allowed to fall freely, how many possible outcomes are likely to occur?

A. 4
B. 8
C. 16
D. 32
E. None of the above is correct.

(D) 8. Each face of two regular tetrahedrons is painted a particular color, red, blue, yellow, and green. If the tetrahedrons are rolled what is the chance that both solids will stop on the white faces?

A. $\frac{1}{2}$
B. $\frac{1}{4}$
C. $\frac{1}{8}$
D. $\frac{1}{16}$
E. None of the above is correct.

(C) 9. To determine the total number of possible outcomes for games of chance, such as tossing coins, the following formula may be used,

A. $\frac{P}{B}$
B. $T = AB$
C. $T = S^n$
D. $T = 2S^n$
E. None of the above is correct.

(C) 10. In a city in which all streets run either north and south or east and west, a person lives three blocks south and two blocks west of his friends. How many different paths along streets can he take from his house to his friends? (Assume they both live on corners.)

A. 3
B. 2
C. 6

D. 4

E. None of the above is correct.

III. Completion Questions.

Find The Measure Of The Chance That These Statements Are True.

($\frac{1}{4}$) 1. If two pennies are tossed, exactly two tails show.

($\frac{3}{4}$) 2. If two pennies are tossed, at least one tail shows.

($\frac{1}{8}$) 3. If three pennies are tossed, exactly three tails show.

($\frac{1}{2}$) 4. If three pennies are tossed, at least two heads show.

For Each Of The Following Exercises, Choose The Event Most Likely To Happen By Comparing Their Measures Of Chance. Show The Measure of Chance For Each Statement.

($\frac{1}{2}$) 5. A. If two coins are tossed, exactly 1 head shows.

($\frac{3}{4}$) B. If two coins are tossed, at least 1 tail shows.

($\frac{1}{2}$) 6. A. If three coins are tossed, at least two tails show.

($\frac{7}{8}$) B. If three coins are tossed, at least 1 head shows.

($\frac{2}{3}$) 7. A. If two cards out of three numbered 2, 3, 4 are chosen without looking, the sum will be odd.

($\frac{1}{3}$) B. If two cards out of three numbered 3, 4, 5 are chosen without looking, the sum will be even.

IV. Computation Questions.

Compute The Measures of Chance For These Events.

($\frac{1}{3}$) 1. A box contains a red marble, green marble and yellow marble. If you draw two marbles without looking, what are the chances that you draw the red and green marble in any order.

($\frac{2}{3}$) 2. Three jackets are in a dark closet. One jacket belongs to you, one to Bill, and one to Bob. If you reach into the closet and take out two jackets, what is the chance that you get your own jacket?

$(\frac{1}{6})$ 3. What is the chance of drawing A B in that order from the box in the example above?

$(\frac{1}{6})$ 4. A die has six faces with 1, 2, 3, 4, 5, or 6 dots on each face. What is the measure of chance that a 6 will turn up when the die is tossed?

$(\frac{1}{6})$ 5. What is the measure of chance of tossing a total of 7 with two dice?

$(\frac{1}{12})$ 6. A die has 12 faces with 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12 dots on each face. What is the measure of chance that a 7 will turn up when the die is tossed?